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on his 62th Birthday

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is being dedicated to honour

Professor S. A. Naimpally

on his 62th Birthday

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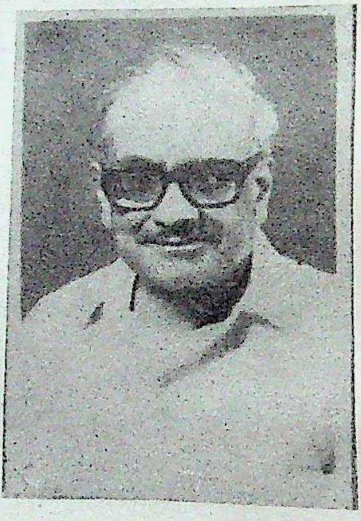

PROFESSOR SOM NAIMPALLY

Jñānābha, Vol. 23, 1993

(Dedicated to Professor S. Naimpally on his 62th Birthday)

SOM NAIMPALLY AND UNIFORM MATHEMATICS

By

S.L. Singh

Gurukula Kangri University, Hardwar - 249404, U.P.

In this article we will have a glimpse of the life and work of Professor Somashekhar Amrith Naimpally – a remarkable man, a towering mathematician, a major topologist and a strong advocate of Advait philosophy, i.e., the theory of one-ness. Som, the ocean of generosity has inspired and helped many students, teachers and colleagues to accept multidimensional challenges of physical reality. It is a modest tribute at the time of his retirement from regular teaching at the age of 62.

Life and Struggle

Somashekhar Amrith Naimpally was born on 31 August 1931 in Bombay. He passed the matriculation examination of the Bombay University in 1948 getting top marks in Mathematics in his school. With equal interest in Mathematics and Physics, he chosed Mathematics, mainly, due to poor eyesight, which made Physics laboratory work difficult. He passed the B.Sc. (with Mathematics principal and Physics subsdiary) in first class with distinction in 1952, M.Sc. 'A' Pure Mathematics (with special papers in Complex Analysis) in first class in 1954 and M.Sc. 'B' Applied Mathematics (with special papers in Analytical Dynamics) in first class in 1958. In each of the above examinations of the Bombay University, he got first rank and in M.Sc. 'B' he broke all previous records.

He began his teaching career in Ruparel College, Bombay as a tutor in 1952 and became a lecturer in 1954. From the very beginning he had repoorts with mathematics students and loved teaching as a profession. Fortunately, he had the benefit of association with great teachers such as Professors N. H. Phadke, M. L. Chandratreya and D. P. Patravali who were colleagues at Ruparel Colleage. Moreover, Professor Phadke had taught him the first two years' college Mathematics and was a major influence in his life. Inspite of his deep love for and good success in teaching, he was quite dissatisfied in not being able to do research. This led him to an 8-month stint as a Senior Research Assistant in Mechanical Engineering at IIT, Kharagpur (October 1958 to May 1959). The red tape and rigid attitude of the then head of the Mechanical Engg. Department did not allow him to pursue research in Turbulence Theory under Professor S.D. Nigam, who was at that time in the Department of Naval Architecture. Reluctantly he returned to Bombay as the head of Mathematics Department at Kirti College. Up to this point, he had no intention to go abroad. Out of sheer frustration in finding himself without research opportunities, he started applying to American universities.

In 1961, he went to Michigan State University as a Teaching Assistant working for the Ph.D. He was fortunate in getting a Fulbright Travel Grant. In 1964, he got his Ph.D. under the well known topologist Professor J. G. Hocking. For an year, he worked at Iowa State University, where he had the benefit of advice from Professor D. E. Sanderson; earlier Professor Sanderson was a visiting professor at Michigan State and had a major influence on him. In 1965, he went to University of Alberta. He returned to India as Professor of Mathematics at IIT, Kanpur in 1969. Due to educational problems of his children, he returned to Canada -- Lakehead University as a professor, where he worked from 1971 to 1988. In 1988, he went to Kuwait to work with Professor C. M. Pareek, whom he knew from his student days at Alberta. He intends to retire and devote himself to full time research/study in July/August 1993.

He is fortunate in having Sudha (Karnad), a gracious and ethical lady, as his life partner. They were married in 1955 and have two sons and a daughter, who have a good appreciation for the finer aspect of life. Both sons play excellent tabla and the daughter is a Bharat Natyam dancer. Sudha is an occupational therapist as well as an accomplished bhajan (devotional songs) singer. In addition, she is adept in several art forms such as painting, sculpturing, sewing etc.

Work

His special interests are in Analysis and Topology in which subject he has over 90 publications -- some of them have appeared in well known journals such as Transactions of Amer. Math. Soc., Proceedings of Amer. Mathe. Soc., Pacific J. Math., Fund. Math., Bull. Austral. Math Soc., Jour. London Math. Soc. etc. He has writen a text book of Trigonometry [91] and co-authored three research-level monographs [92-94] one of which is Proximity Spaces published in the famous series of Cambridge Tracts [93]. His foresight in co-authoring a monographs [92]-on Quasi-uniform Spaces in 1965 was recently Justified (in 1991) when a special conference was held in this subject in Berne, Switzerland. This subject has become very important in theoretical computing.

In his Ph.D. Thesis "Essential fixed points and almost continuous functions" (1964), he discussed a new function space topology called Graph Topology (see [5]. This topology is not only useful in Topology but also in Approximation Theory, Analysis and Differential Topology. Several mathematicians have worked on it, e.g., Beer Poppe, Hansart, Pareek. Then he worked on quasi- uniform spaces on which he wrote a monograph [92] with Murdeshwar (1966). This gave a boost to research on this topic, which has now become remarkably important due to its applications to theoretical computing. His monograph [93] published in collaboration with his master's student Warrack appears to be the only book and a standard reference on Proximity Spaces.

Subsequently, most of his research was in proximity spaces, nearness spaces and applications [17, 18, 21-25, 27 and others].

He also worked on open mpping theorem [38], convexity (solving a major problem) [40].

Recently he has been working on Hyperspaces in which he has brought in proximity [82]. He has also combined proximity and graph topology in [90].

In addition to the above, he has interest in contractive and anti-contractive mappings, fixed point theorems [2, 3, 47-55, 65], non-continuous functions etc. Hybrid contractive and hybrid nonexpansive mappings are studied for the first time in [65]. Quite recently, fixed point theorems for such mappings have been found useful in Pareto type optimization study. He has studied ancient Indian Mathematics, contributed a few papers in this area, and has been the main architect behind the English rendering of Bhāskārāchārya' s Līlavatī [95]. He, also assisted by his gracious better-half, devoted a lot of time and spent a good amount of his personal money to complete this monumental work, and has very humbly declined its royalities.

Talks, visits and ... ...

He has given invited talks at the Indian Mathematical Society at Gorakhpur in 1970; Spring Topology Conferences in Oklahoma 1972, Memphis 1975, Wisconsin 1991; Italian Topology Conferences in Taormina 1984, Sorrento 1988; Top. Conf. in Paola, Brazil 1988; Top. Conf. in Madison, Wisconsin, 1992; and several others. He has been a member of American Math. Soc., Indian Math. Soc. Italian Math. Soc., Canadian Math. Soc., Indian Math. Soc., Italian Math. Soc., Canadian Math. Soc. and Mathematical Association of America. He has given colloquium talks at several universities in India, U.S.A. , Canada, Italy, Japan and Kuwait.

Som was a visiting professor at Southern Illinois University, IIT, Bombay, Universities of Salerno, Napoli, Torino; Meerut University, etc. He has been a distinguished visitor at California State University (1988) and Univ. of Saugar (1971).

Besides Mathematics, he is interested in sports, music, philosophy, biographies and organizing, classical dances. He played cricket for his school and was a champion in Table Tennis in Ismail College in 1950. He was a badminton champion in Michigan State University, 1964 (in mixed doubles with his life partener Sudha as partner) and at Iowa State University in 1965. He plays a little tabla and has liatened to a large number of top Hindustani classical musicians, many of whom have been his guests at home. He started a Music Circle at IIT, Kanpur which became a great success and he has helped very generously several musicians and dancers during their visits to Canad / U.S.A. .

Postretirement Plans

Som's on - going research centres around hyperspaces, function spaces and extensions of functions. In his April 10, 1993 communication, he writes: "During the year 1990-91 when Kuwait was occupied, I received an honorary appointment from Carleton University. There are no teaching dutis and [after retirement from Kuwait University] I hope to be able to move freely on research and lecturing assignments."

RESEARCH AND PUBLICATIONS

Papers

1. A new proof for a theorem of classical dynamics, *Math. Student* **28** (1960), 93-94.
2. A note on contractive mappings, *Indag. Math.* **26** (1964), 275-279.
3. Contractive mappings in uniform spaces, *Indag. Math.* **27** (1965), 477-481.
4. A generalization of the Kuratowski embedding theorem, *Nieuw Archief Voor Wiskunde* (3) **13** (1965), 113-115.
5. Graph topology function spaces, *Transactions Amer. Math. Soc.,* **123** (1966), 267-272.
6. Function spaces of invertible spaces, *Amer. Math Monthly* **73** (1966), 513-515.
7. On strongly continuous functions, *Amer. Math. Montly* **74** (1967), 166-168.
8. Separation axioms for quasi-uniform spaces, Amer. Math. Monhly, 74(1967), 283-284.
9. Function spaces of quasi-uniform spaces, *Indag Math.* **27** (1965), 768 - 771.
10. Function space topologies for connectivity and semi-connectivity functions, *Canad. math. Bull.* **9** (1966), 349-352.
11. On semi-Hausdorff spaces [with M.G. Murdeshwar], *Canad. Math. Bull.* **9** (1966), 353-356.
12. Connected-open topology for function spaces [with A. Irudayanathan], *Indag. Math.* **28** (1966), 22-24.
13. Trennungsaxioms in quasi-uniform spaces, *Niew Arch. Wiskunde* (3), **14** (1966), 97-101.
14. R_1 - topological space [with M.G. Murdeshwar], *Canad. Math. Bull.* **9** (1966), 521-523.
15. On uniform completion [with C. M. Pareek], *Indag. Math.* **29** (1967), 123-125.
16. On R_0-topological spaces, *Ann. Univ. Budapest Eotvos Sec.* Math. **10** (1967), 53-54.
17. Clusters and ultrafilters [with B. Warrack], *Publ. I'Inst. Math.* 8 (22), (1968), 100-101.
18. On completion of a uniform space by local clusters [with B. Warack], *Proc. Int. Symp. Ex[illegible] th., Berlin* (1967), 239-241.
19. Hyperspaces of invertible spaces, *Math. Student* **35** (1967), 39-41.
20. Graph topologies for function spaces II (with C.M. Pareek) *Prace Matem.* **13** (1970), 227-237.
21. Proximity approach to extension problems, [with M.S. Gagrat] *Fund. Math.* **71**(1971) 63-76.

22. Wallaman compactifications and Wallman real compactifications [with M.S. Gagrat], *J. Austr. Math. Soc.* **15** (1973), 417-427.
23. Construction of Lodato proximities [with P. L. Sharma], *Math. Japonicae* **15**(1970), 101-104.
24. On two problems of Harris concerning R-proximities, [with P.L. Sharma], *Proc. Amer. Math. Soc.* **32**(1972), 312.
25. Proximity approach to extension problems II, [with M.S. Gagrat], *Math. Jponicae* **16** (1971), 35-43.
26. On almost complete and almost precompact quasi-uniform spaces, [with P. Fletcher], Czech. Math. J. **21** (1971), 283 - 390.
27. Proximity approach to semi-metric and developable spaces, [with M. Gagrat], *Pacific J. Math.* **44** (1973), 93-105.
28. Some pretopologies associated with a given topology [with K.M. Garg], *Proc. Third Prague Conf.* (1971), 145-146.
29. Hausdorff compactifications as eepireflections [with W.N. Hunsaker], *Canad. Math. Bull.* **17** (1975), 675-677.
30. Wallman, T_1 - compactifications as epireflections [with H.L. Bentely], *Gen. Top. Appl.* **5** (1974), 29-41.
31. L-real compactifications as epireflections [with H.L. Bentley], *Proc. Amer. Math. Soc.* **44** [1974), 196-202.
32. A counter example concerning almost continuous functions, *Proc. Amer. Math. Soc.* **43** (1974), 475.
33. Reflective functors via nearness, *Fund. Math.* **85** (1974), 245-255.
34. Nearness : a better approach to continuity and limits [with P.C. Cameron and J.G. Hocking], *Amer. Math Monthly* **81** (1974) 739-745.
35. Characterization of a convex subspace of linear topological space [with M.D. Guay], *Math. Japonicae* **20** (1975), 37-41.
36. Local compactness of families of continuous pointcompact relations [with W.N. Hunsaker], *Pacific J. Math.* **52** (1974), 101- 105.
37. Not every near family is contained in a clan [with J.H.M. Whitfield], *Proc. Amer. Math. Soc.* **47** (1975), 237-238.
38. Open and uniformly open relations [with P. Mah], *Bull. Amer Math. Soc.* 80[1974), 1157-1159, *Proc. Amer. Math. Soc.* **66** (1977), 159- 166.
39. Applications of proximity spaces to topology (invited address), *Math. Student* **39** (1971), 235-237.
40. Linearization of a convexity space (with P. Mah and J. H. M. Whitfield], *J. London Math. Soc.* (2) **13** (1976), 209-214.
41. Nearness in Topology and Elsewhere, (Invited address), Topology *Proc., Marcel Dekker* (1976), 77-86.
42. Functionally determined proximities and uniformities (Invited address), *Topology Conf. Oklahoma* (1972), 234-240.
43. A generalization of a theorem of Marjanovic and Kiang concerning non-expansive maps [with R. Anantharaman], *Indag. Math.* **39** (1977), 1-3.
44. Boundedness and equicontinuity of measures and linear functionals [with R. Anantharaman], *Math. Sem. Notes* **7** (1979), 271-285.
45. An abstract model for measure and category [with S.M.F. Ho], *Math. Japonicae* **24** (1979), 123-140.
46. On T1-compactifications [with M.L. Tikoo], *Pacific J. Math* **84(1979), 183-190.**
47. Fixed points and close -to -normal structure [with K. L. Singh and J.H. M. Whitfield], Nonlinear Analysis, *Appl.. Marcel Dekker* (1982), 203-221.
48. Fixed points and sequences of iterates in locally convex spaces [with K. L. Singh and J. H. M. Whitfield], Top. Methods in Nonlinear Functional Analysis, Amer.

Math. Soc. **21** (1983), 159- 166.

49. Sequences of iterates in locally convex spaces [with K. L. Singh], Nonlinear Phenomena in Mathematical Sciences, *Academic Press* (1982), 725-736.
50. Extensions of some fixed point theorems of Rhoades [with K. L. Singh] *J. Math. Anal. Appl.* **96** [1983), 437-446.
51. Common fixed points for nonexpansive and asympotically nonexpansive mappings [K. L. Singh and J. H. M. Whitfield] *Comm. Math. Univ. Carolina* **24** (1983), 281-300.
52. Fixed points and nonexpansive retracts in locally convex spaces, [With K.L. Singh and J.H.M. Whitfield], *Fundamenta Math.* **120** (1984), 63-75.
53. Common fixed point theorems [with K. L. Singh and J. H. M. Whitfield], *Top. Proc.* **61** (1981), 201-202.
54. Fixed and common fixed points for nonexpansive mappings in convex metric spaces [with K. L. Singh and J. H. M. Whitfield], *Math. Sem. Notes Kobe Univ.* **11** (1983), 239-248.
55. Fixed points in convex metric spaces [with K.L. Singh and J.H.M. Whitfield], *Math Japan.* **25** (1984), 585-597.
56. Proximal convergence [with A. Di Concilio], *Monat, fur Math.* **103** (1987), 93-102.
57. Wijsman convergence for function spaces, Top. Conf. Trieste (1986), *Rend. Circ. Mat. Palermo* **18** (1988), 343-358.
58. Topological convergence and uniform convergence, *Czech. Math. J.* **37** (112), (1987), 608-612.
59. Best weithted approximation of vector-valued functions [with D.V. Pai], Proc. Int. Conf. 75th Birthday of G.G. Lorentz, St. John's Newfoundaland (1984), Res. Notes Math. **133** (1985), 140-158.
60. Best weighted approximation of vector-valued functions [with D.V. Pai], Proc. Int. Conf. 75th Birthday of G.G Lorentz, St. John's Newfoundland (1984), *Res. Notes Math.* **133**.(1985), 140-158.
61. A unified approach to metrization problems [with A.Di Concilio], *Acta Math. Hung.,* **53** (1989), 109-113.
62. Extrensions of continuous functions : reflective functors [with W. N. Hunsaker], *Bull. Austr. Math. Soc.,* **35** (1987), 455 - 470
63. Local compactness of function spaces [with S.K. Kaul], *Atti Della Academia Della Scienze di Torino* **120** (1986), 48-54.
64. **Nearness :** a better approach to continuity and limits [with P. Cameron and J.G. Hocking], *Math. Collog.* 3 (1985), 6-13; 52-60.
65. Coincidence theorems for hybrid contractions [with *S. L. Singh and J.H. M. Whitfield], Math.* Nachr **127** (1986), 177-180.
66. Fine uniformity and the locally finite hyperspace typology [with P. L. Sharma], *Proc. Amer. Math. Soc.,* **103** (1988), 641 -646.
67. A nearness approach to extension problems [with A Di Concilio], *Acad. di XL,* **11** (1987), 161-171.
68. Atsuji problem for nearly continuous and quasi-continuous functions [with Filippo Cammaroto], Ques. *Ans. in Topology* **7** (1989), 9-13.
69. Assuji spaces : continuity versus uniform continuity [with A. Di Concilio], *Proc. VI Brazilian Conf. Top.* (1988).
70. Totally bounded Moore spaces [with Maria Brigitte Agnello and Giuseppe Di Maio], (submitted).
71. The nearly compact-open hypertopology [with Giuseppe Di Maio], *Atti. Accad. Sci. Torino* **122** (1988), 164-169.
72. Proximity extensions [with G. Di. Maio], *J. Nat Phys. Sci.* **1** (1987), 64-66.
73. Near comapctifications [with Filippo Commaroto], *Math. Nachr.,*
74. Fixed functions and Blumberg sets : a proximity approach [with Giuseppe Di

Maio], (sumbitted).

75. Fixed points of multi-mappings related to minimization problems [with D.V. Pai], Nonlinear Analysis and Appl., *Marcel and Dekker,* (1987), 339-403.

76. A general approach to symmetric proximities [with G. Di Maio], *Top. Conf. Sorrento* (to be published).

77. Remarks on D-completely regular spaces [with G. Di. Maio and C.M. *Pareek], Top. Conf. Sorrento* (to be published).

78. D-proximity spaces (with G. Di Maio], (submitted).

79. D-proximity spaces [with G. Di Maio], (submitted).

80. Preservation of generalized continuity [with G. Di. Maio], (submitted).

81. Preservation of D-complete regularity [with F. Cammaroto], (submitted).

82. Proximinal hypertopologies [with A. Di Concilio and P.L. Sharma], *Proc. VI Brazilian Conf. Top.* (1988), (to appear).

83. Comparison of topologies on hyperspaces, New York Academy of Science (to appear)

84. Characterizations of metric and generalized metric spaces by realvalued functions, Questions and *Answers in General Topology* **8** (1990), 425-439.

85. Hyperspaces and function spaces, *Questions and Answers in General Topology* **9** (1991), 33-60.

86. Generalized metric spaces via annihilators, *Questions and Answers in General Topology* **9** (1991),

87. The locally finite hypertopology and generalized uniformities, *Zbornik Radova* **5** (1991), 113-116.

88. Abstract measure of farness and Wijsman convergence, *Zbornik Radova* **5,** 109-112.

89. Abstract measure of farness and hypertopology, Rend. *Circ. Mat. Palermo* **29** (1992), 329-337.

90. Proximal graph topologies, Qeustions and *Answers in General Topology* **10** (1992), 97-125.

Books

91. S. A. Naimpally, *A New Course in Trigonometry,* Popular Prakasan, Bombay, (1959).

92. M. G. Murdeshwar and S. A. Naimpally, *Quasi-uniform topological spaces,* P. Noordhoff Ltd. (1966), Groningen.

93. S. A. Naimpally and B.D. Warrack, *Proximity Spaces,* Cambridge University Press, U.K. (1970).

94. C. J. Mozzochi, M.S. Gagrat and S. A. Naimpally, *Symmetric Generalized Topological Spaces,* Expositon Press, New York (1976).

95. K. S. Patwardhan, S. A. Naimpally and S. L. Singh, *The Lilavati of Bhaskaracharya* (Ed.), Motilal Banarsidas, Delhi (to appear).

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(Dedicated to Professor S. A. Naimpally on his 62th Birthday)

ON A CLASS OF FUNCTIONS CONNECTED WITH THE HYPER-BESSEL FUNCTIONS

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ABSTRACT

In this paper, we introduce a class of functions of an analogous nature to the hyper-bessel functions treated by P. Delerure [4]. We establish some of their main properties: generating function, recurrence relations, differential equation and certain integral representations.

1. Introduction

The hyper-Bessel function $J^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$ which generalizes the ordinary Bessel function with respect to the number of indices, is defined by the formula

$$J^{(n)}_{\lambda_1, \dots, \lambda_n} = \frac{\left(\frac{x}{n+1}\right)^{\lambda_1 + \dots + \lambda_n}}{\Gamma(\lambda_1 + 1) \dots \Gamma(\lambda_n + 1)} \; {}_0F_n \left[\lambda_1 + 1, \dots, \lambda_n + 1; -\left(\frac{x}{n+1}\right)^{n+1} \right]. \qquad \dots (1.1)$$

It was earlier investigated by Delerue [4].

For $n=1$, (1.1) reduces to the usual Bessel function $J_\lambda(x)$. The case $n=2$, giving rise to the so-called Bessel-function of the third order, was initially considered and studied in a series of works by P. Humbert ([10], [11], [12]) and later on by other authors (R.S. Varma [15], A.K. Agarwal [1], R.P. Agarwal [2]). More recently, I. Dimovski and V. Kiryakova, among other investigators, have obtained interesting results by using the hyper-Bessel functions $J^{(n)}_{\lambda_1, \ldots, \lambda_n}(x)$ in relation with the general Bessel- type differential operator

$$B = t^{\alpha_0} \frac{d}{dt}\ t^{\alpha_1} \frac{d}{dt} \ldots t^{\alpha_{n-1}} \frac{d}{dt}\, t^{\alpha_n}$$

(see, for example, [5], [13], [14]), which appears in important differential equations of mathematical physics (in particular, the second order singular differential operator

$$B = x^{-2}\,(x \frac{d}{dx} + \nu)\,(x \frac{d}{dx} - \nu)$$

occurring, e.g., in the heat equation and in some partial differential equations of degenerated or mixed type).

In the present paper we introduce a class of functions of an analogous nature to (1.1) and investigate some of their basic properties. For such functions, we write

$$C^{(n)}_{\lambda_1, \ldots, \lambda_n}(x)$$

and they will be called "Bessel-Clifford functions of order n". For the sake of simplicity, we sometimes shall write $C_{\lambda_1, \ldots, \lambda_n}(x)$ instead of $C^{(n)}_{\lambda_1, \ldots, \lambda_n}(x)$.

The case n=1 leads to the Bessel-Clifford funtion $C^{(x)}_{\lambda}$ studied in detail by N. Hayek [6]. The importance of this funtion lies mainly in the fact that it substitutes $J_\lambda(x)$ advantageously in a great deal of theoretical and practical contexts (see [6] and references therein).

The functions $C_{\lambda_1, \lambda_2}(x)$ (corresoponding to n=2) were initially considered and treated by N. Hayek [7]. Later on, other several important properties of these functions were also analyzed by N. Hayek and V. Hernandez ([8], [9]).

Following a similar approach as that followed by Delerue for the hyper-Bessel functions [4], we make here a systematic study of the functions $C^{(n)}_{\lambda_1, \ldots, \lambda_n}(x)$. We otbatin their generating function, recurrence formulas, the relation with the hyper-Bessel functions, and the differential equation of order $n+1$ satisfied by them.

Furthermore, some integral representations for $C^{(n)}_{\lambda_1, \ldots, \lambda_n}(x)$ are also established.

On the other hand, it is necessary to point out that the kind of functions here introduced, have a simpler analytical structure than the Delerue hyper-Bessel functions. Therefore, we feel that the fucntions

$$C^{(n)}_{\lambda_1, \ldots, \lambda_n}(x)$$

can be usefully exploited in many theoretical and applicative areas in which Delerue's functions and related ones appear.

In this paper, all the parameters are taken to be real unless otherwise stated.

2. The Function $C^{(n)}_{\lambda_1, \lambda_2, \dots, \lambda_n}(x)$

The Bessel-Clifford function $C_\lambda(x)$ has been studied in a series of works by Hayek (see mainly [6]).

First, we recall that this function, whose manifold properties can be seen in [6], possesses the series representation

$$C_\lambda(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{\Gamma(\lambda + r + 1)\, r\,!} \tag{2.1}$$

and admits $e^{u - x/u}$ as a generating function.

It is the principal solution of the linear differential equation $x y'' + (\lambda + 1) y' + y = 0$, and is related with the usual Bessel fucntion by means of the equality :

$$C_\lambda(x) = x^{-\lambda/2} J_\lambda(2\sqrt{x}). \tag{2.2}$$

This function may be generalized with respect to the number of indices yielding

$$C^{(n)}_{\lambda_1, \dots, \lambda_n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{\Gamma(\lambda_1 + r + 1) \dots \Gamma(\lambda_n + r + 1)\, r\,!} \tag{2.3}$$

From this series representation, it is evident that

$$C^{(n)}_{\lambda_1, \dots, \lambda_n}(x) = \frac{1}{\Gamma(\lambda_1 + 1) \dots \Gamma(\lambda_n + 1)}\, {}_0F_n(\lambda_1 + 1, \dots, \lambda_n + 1; -x) \tag{2.4}$$

where ${}_0F_n(\lambda_1 + 1, \dots, \lambda_n + 1; -x)$ is the well-known hypergeometric function.

Henceforth, $C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$ will be called "the Bessel-Clifford function of the order n and indices $\lambda_1, \dots, \lambda_n$".

From (2.3) it follows that $C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$ is an analytic function of x for all x, and an analytic function of λ_i for all λ_i.

Since (2.3) converges absolutedly and uniformly on any bounded interval, the series can be differentiated or integrated term by term, with reppect to x or to λ_i.

From (1.1) and (2.4), we obtain

$$C^{(n)}_{\lambda_1, \dots, \lambda_n}(x) = x^{-(\lambda_1 + \dots + \lambda_n)/(n+1)}\, J^{(n)}_{\lambda_1, \dots, \lambda_n}\left((n+1)\, x^{1/(n+1)}\right). \tag{2.5}$$

Clearly, the definition (2.4) makes sense only when the indices λ_i are not negative integers. To avoid this inconvenience, we take as definition the following generating function of n integral parameters $m_1, \dots, m_n$:

$$e^{u_1 + \dots + u_n - x/(u_1 \dots u_n)}$$

$$= \sum_{m_1 = -\infty}^{+\infty} \dots \sum_{m_n = -\infty}^{+\infty} u_1^{m_1} \dots u_n^{m_n} \, C_{m_1, \dots, m_n}(x). \tag{2.6}$$

Thus, if the integers m_1 are all positive, the series (2.6) leads us agian to (2.4). If some of these m_i are negative, then we can suppose that they are increasingly ordered, i.e.,

$$-m_1, \dots -m_k, +m_{k+1}, \dots, +m_n,$$

and after the change of variable in the generating function

$$u_1 \dots u_n = x/t\,,$$

we have the following relation :

$$C_{-m_1, \dots -m_k, +m_{k+1}, \dots, +m_n}(x) =$$

$$= (-x)^{m_1} \, C_{m_1, m_1 - m_2, \dots, m_1 - m_k, m_1 + m_{k+1}, \dots, m_1 + m_n}(x) \tag{2.7}$$

Analogously, we can express other similar formulas in which the right-hand memebrs include powers of $(-x)$ with other subindices as exponents. All of these relations allow us to define the Bessel Clifford functions of the order n, in the cases in which some indices are negative integers and the rest non integer positive numbers.

3. Recurrence Relations and Differential Formulas with Respect to the Indices

The symbolic calculus of n variables, used by Delerue [3] to study the properties of the hyper-Bessel functions, can be also employed to establish several recurrence formulas of different types for the $C^{(n)}_{\lambda_1, \dots, \lambda_n}(x)$.

Let us consider the following general symbolic relation [3] :

$$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \dots x_n^{\alpha_n} f(x_1, x_2, \dots, x_n) \underset{n}{\supset}$$

$$(-1)^{\alpha_1 + \dots + \alpha_n} p_1 \dots p_n \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial_{p_1}^{\alpha_1} \dots \partial_{p_n}^{\alpha_n}} \left[\frac{\varphi(p_1, \dots, p_n)}{p_1 \dots p_n} \right] \tag{3.1}$$

in which $x_1, \dots, x_n$ correspond to the symbolic variables $p_1, \dots, p_n$, respectively.

Then, starting from the function of n variables :

$$\frac{x_1^{\lambda_1} \dots x_n^{\lambda_n}}{(\Gamma(\lambda_1 + 1) \dots \Gamma(\lambda_n + 1))} \, {}_0F_n(\lambda_1 + 1, \dots, \lambda_n + 1; x_1 \dots x_n)$$

and, after some manipulation, we get

$$x_1^{\lambda_1} \dots x_n^{\lambda_n} C_{\lambda_1, \dots, \lambda_n}(x_1 \dots x_n) \underset{n}{\supset}$$

$$\underset{n}{\supset} \quad \frac{1}{p_1^{\lambda_1} \dots p_n^{\lambda_n}} \, e^{-1/(p_1 \dots p_n)} \tag{3.2}$$

Let us now consider the more simple case in (3.1), i.e.

$$x_1 f(x_1, \dots, x_n) \underset{n}{\supset} - p_1 \frac{\partial}{\partial p_1} \left[\frac{\varphi(p_1, \dots, p_n)}{p_1} \right]$$

then, taking into account (3.2), the following recurrence relation, holds :

$$C_{\lambda_1 - 1, \lambda_2, \dots, \lambda_n}(x) + x \, C_{\lambda_1 + 1, \dots, \lambda_n + 1}(x) = \lambda_1 C_{\lambda_1, \dots, \lambda_n}(x) \, . \tag{3.3}$$

Thus, summing all the analogous relations to the (3.3), we can write :

$$\begin{aligned} & C_{\lambda_1 - 1, \lambda_2, \dots, \lambda_n}(x) + C_{\lambda_1, \lambda_2 - 1, \dots, \lambda_n}(x) + \dots + \\ & \quad + C_{\lambda_1, \lambda_2, \dots, \lambda_n - 1}(x) + n \, x \, C_{\lambda_1 + 1, \dots, \lambda_n + 1}(x) = \\ & \quad = (\lambda_1 + \dots + \lambda_n) \, C_{\lambda_1, \dots, \lambda_n}(x) \end{aligned} \tag{3.4}$$

Similarly, by particularizing the left-hand member of the general symbolic relation (3.1) to the case $x_1 x_2 f(x_1, \dots, x_n)$,

it is also inferred :

$$\begin{aligned} C_{\lambda_1, \dots, \lambda_n}(x) = {} & x^2 \, C_{\lambda_1 + 3, \lambda_2 + 3, \lambda_3 + 2, \dots, \lambda_n + 2}(x) + \\ & + (\lambda_1 + 1)(\lambda_2 + 1) \, C_{\lambda_1 + 1, \lambda_2 + 1, \lambda_3, \dots \lambda_n}(x) \\ & - (\lambda_1 + \lambda_2 + 3) \, x \, C_{\lambda_1 + 2, \lambda_2 + 2, \lambda_3 + 1, \dots, \lambda_n + 1}(x). \end{aligned} \tag{3.5}$$

Moreover, when the particular symbolic relation

$$x_1 \frac{\partial f}{\partial x_1} \supset - p_1 \frac{\partial \varphi}{\partial p_1}$$

is applied to (3.2), one has :

$$C^{(p)}_{\lambda_1, \lambda_2, \dots, \lambda_n}(x) = - C_{\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_{n+1}}(x). \tag{3.6}$$

Repeatedly differentiation p times, it follows :

$$C^{(p)}_{\lambda_1,\dots,\lambda_n}(x) = (-1)^p\, C_{\lambda_1+p,\dots,\lambda_n+p}(x). \tag{3.7}$$

From (3.3) and (3.6), it also results :

$$\lambda_1 C_{\lambda_1,\dots,\lambda_n}(x) + C'_{\lambda_1,\dots,\lambda_n}(x) = C_{\lambda_1-1,\lambda_2,\dots,\lambda_n}(x)\ . \tag{3.8}$$

Furthermore, we can obtain :

$$\frac{d}{dx}\left[x^{\lambda_1} C_{\lambda_1,\dots,\lambda_n}(x)\right] = x^{\lambda_1-1} C_{\lambda_1-1,\lambda_2,\dots,\lambda_n}(x) \tag{3.9}$$

which, by using (3.2), may be generalized without difficulty to the case of n variables $x_1, \dots, x_n$.

The sum of similar relations to (3.8) and (3.9), respectively, yield other formulas.

On the other hand, differentiating (3.8) with respect to x, and after some algebra, we have :

$$C''_{\lambda_1,\dots,\lambda_n}(x) + \frac{\lambda_1+\lambda_2+1}{x} C'_{\lambda_1,\dots,\lambda_n}(x) + \\ + \frac{\lambda_1\lambda_2}{x^2} C_{\lambda_1,\dots,\lambda_n}(x) = \frac{1}{x^2} C_{\lambda_1-1,\lambda_2-1,\lambda_3,\dots,\lambda_n}(x)\ . \tag{3.10}$$

A new differentitaion also gives :

$$C'''_{\lambda_1,\dots,\lambda_n}(x) + \frac{\lambda_1+\lambda_2+\lambda_3+1}{x} C''_{\lambda_1,\dots,\lambda_n}(x) + \\ + \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3+1}{x^2} C'_{\lambda_1,\dots,\lambda_n}(x) + \\ + \frac{\lambda_1\lambda_2\lambda_3}{x^3} C_{\lambda_1,\dots,\lambda_n}(x) = \\ = \frac{1}{x^3} C_{\lambda_1-1,\lambda_2-1,\lambda_3-1,\lambda_4,\dots,\lambda_n}(x) \tag{3.11}$$

Iterating the process, we obtain the form of the more general relation including the derivative of the k^{th} order of the function $C_{\lambda_1,\dots,\lambda_n}(x)$.

Remark 1. For n=1 and 2, we get again recurrence relations satisfied by $C_\lambda(x)$ and $C_{\lambda_1,\lambda_2}(x)$. For example, (3.6) and (3.7) are reduced to formula (6.1) ([6]) and (3.2), (3.3) ([7]), respectively.

Remark 2. The relations (3.10), (3.11) and the subsequent ones allow to calculate through the corresponding differential equations, the function $C_{\lambda_1,\dots,\lambda_n}(x)$ by means of :

$$C_{\lambda_1-1,\lambda_2-1,\lambda_3,\dots,\lambda_n}(x),\ C_{\lambda_1-1,\lambda_2-1,\lambda_3-1,\lambda_4,\dots,\lambda_n}(x),$$

etc., respectively.

Thus, for example, from (3.10) it results the differential equation :

$$x^2 y'' + (\lambda_1 + \lambda_2 + 1)\, x\, y' + \lambda_1 \lambda_2\, y = = C_{\lambda_1 - 1, \lambda_2 - 1, \lambda_3, \dots, \lambda_n}(x)$$

which can be solved by the well-known methods.

Next, we will establish for $C_{\lambda_1, \dots, \lambda_n}(x)$ some differentiation formulas with respect to the indices.

Using a method by Delerue [4] (generalization of the Humbert's one [11]) for the hyper-Bessel functions, we start differentiating the following symbolic relation with respect to λ_1 :

$$x^{\lambda_1} \log x\, C_{\lambda_1, \dots, \lambda_n}(x) + x^{\lambda_1} \frac{\partial}{\partial \lambda_1}\left[C_{\lambda_1, \dots, \lambda_n}(x)\right] \supset$$
$$\supset -p^{-\lambda_1} \log p\, C_{\lambda_1, \dots, \lambda_n}(1/p)$$

Then, by applying the composition product theorem and taking into account the relation which is deduced integrating (3.9), it is inferred :

$$\frac{\partial}{\partial \lambda_1}\left[C_{\lambda_1, \dots, \lambda_n}(x)\right] = \gamma\, C_{\lambda_1, \dots, \lambda_n}(x) +$$
$$+ \frac{1}{x} \int_0^x \left[\log\left(1 - \frac{y}{x}\right)\right]\left(\frac{y}{x}\right)^{\lambda_1 - 1} C_{\lambda_1 - 1, \lambda_2, \dots, \lambda_n}(y)\, dy \qquad (3.12)$$

where γ is the Euler constant. Other similar formulas can be deduced corresponding to the remaining values of the indices.

If all the indices are equal to m, on differentiating with respect to m the symbolic relation deduced from (3.2) by putting all the λ_1 equal to m, we have, after using several symbolic calculus rules and some transformations :

$$\frac{\partial}{\partial m}\left[C_{m, \dots, m}(x)\right] = n\, \gamma\, C_{m, \dots, m}(x) +$$
$$+ n \int_0^1 \left[\log(1 - t)\right] t^{m-1}\, C_{m-1, m, \dots, m}(tx)\, dt \qquad (3.13)$$

A similar result is inferred when only k of the indices were equal to m.

Remark 3. Note the more simplified forms of the abvoe formulas with respect to the ones obtained by Delerue [4] for the $J_{\lambda_1, \dots, \lambda_n}(x)$.

In particular, (3.12) it reduces for the third-order Bessel-Clifford fucntion to the formula :

$$\frac{\partial}{\partial m}\left[C_{m, n}(x)\right] =$$

$$\gamma C_{m,n}(x) + \frac{1}{x} \int_0^x \left[\log\left(1 - \frac{y}{x}\right)\right]\left(\frac{y}{x}\right)^{m-1} C_{m-1,n}(y)\,dy$$

Identical expression holds for $C_m(x)$.

4. Differential Equation

In the previous section we have concluded that the function $C_{\lambda_1,\ldots,\lambda_n}(x)$ satisfied a differential equation of the form :

$$x^n \frac{d^n y}{dx^n} + \alpha_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + \alpha_{n-k} x^k \frac{d^k y}{dx^k} + \ldots +$$

$$+ \alpha_{n-1} x y' + \alpha_n y = C_{\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{n-1}}(x) \tag{4.1}$$

The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$, are the same as those obtained by Delerue for the $(n+1)^{th}$ order differential equation satisfied by the hyper- Bessel functions of the order n, i.e. $\alpha_n = \sigma_n$

$$\alpha_n + \alpha_{n-1} = 1 + \sigma_1 + \ldots + \sigma_n,$$

$$k!\,\alpha_{n-k} + \frac{k!}{1!}\alpha_{n-k+1} + \frac{k!}{2!}\alpha_{n-k+2} + \ldots + \quad + \frac{k\,!}{(k-1)!}\alpha_{n-1} + \frac{k!}{k!}\alpha_n =$$

$$= k^n + k^{n-1}\sigma_1 + k^{n-2}\sigma_2 + \ldots + k^{n-k}\sigma_k + \ldots + k\,\sigma_{n-1} + \sigma_n \tag{4.2}$$

In particular, $\alpha_1 = \sigma_1 + \frac{n(n-1)}{2}$. The quantities $\sigma_1, \sigma_2, \ldots, \sigma_n$, denote the sum of the partial products obtained by multiplying 1,2,..,k,..n coefficients $\lambda_1, \ldots, \lambda_k, \ldots, \lambda_n$.

Now, differentiating (4.1) and taking into account (3.6), one deduces the following linear differential equation of the order n+1 verified by $C_{\lambda_1,\lambda_2,\ldots,\lambda_n}(x)$.

$$x^{n+1} y^{(n+1)} + \left\{\sigma_1 + \frac{n(n+1)}{2}\right\} x^n y^{(n)} + \ldots + \left\{k\,\alpha_{n-k} + \alpha_{n-k+1}\right\} x^k y^{(k)} +$$

$$+ \ldots + \left\{\alpha_{n-1} + \alpha_n\right\} x\,y' + x\,y = 0 \tag{4.3}$$

k taking the values n –1, n– 2, ..., 3, 2, 1.

Remark 4. For n=1, we obtain the known differential equation satisfied by $C_{\lambda_1,\lambda_2}(x)$ [8].

Theorem. If the indices $\lambda_1, \lambda_2, \ldots, \lambda_n \notin Z$ or $\lambda_i - \lambda_j \notin Z$ $(\forall \neq j)$, the differential equation (4.3) has as fundamental system of solutions, the following n+1 functions :

$$C_{\lambda_1,\lambda_2,\ldots,\lambda_n}(x),\ x^{-\lambda 1} C_{-\lambda_1, \lambda_2 - \lambda_1, \ldots, \lambda_n - \lambda_1}(x),$$

$$x^{-\lambda_2} C_{-\lambda_2, \lambda_1 - \lambda_2, \ldots, \lambda_n - \lambda_2}(x), \ldots, x^{-\lambda_n} C_{-\lambda_n, \lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n}(x)$$

(4.4)

Proof : The $(n+1)$ linearly independent solutions (4. 4) of (4. 3), can be easily obtained by applying the well-known Frobenius method.

5. Integral Representations of $C_{\lambda_1, \lambda_2, \ldots, \lambda_n}(x)$

Starting from the definition of the generalized hypergeometric function introduced by C.S. Meijer as a Mellin-Barnes type integral [13], i.e.

$$G^{m.n}_{p,q}(z) = G^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} a_1, & a_2, \ldots, a_p \\ b_1, & b_2, \ldots, b_q \end{matrix}\right]$$

we can establish an interesting integral representation for the treated fucntions, namely :

$$C^{(n-1)}_{\lambda_1, \ldots, \lambda_{n-1}}(z)$$

$$= \left[\frac{n}{(2\pi)^{n-1}}\right]^{\frac{1}{2}} \int_0^1 G^{n-1,0}_{n-1,n-1}\left[t \,\middle|\, \begin{matrix} \lambda_1, \ldots, \lambda_{n-1} \\ \frac{1}{n}-1, \frac{2}{n}-1, \ldots, \frac{n-1}{n}-1 \end{matrix}\right] \cos_n\left[n(zt)^{1/n}\right] dt$$

with $\lambda_k > \frac{k}{n} - 1, \ k = 1, 2, \ldots, n-1$

and $\cos_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{nj}}{(nj)!}$

being the generalized cosine function of order $n > 1$ (see [5]).

Note that the formula (5.1) is analogous to the one obtained by Dimovski and Kiryakova [5].

Remark 5. For n=2 and $\lambda_1 = \nu > -\frac{1}{2}, \lambda_2 = 0$ the formula (5.1) reduces to the integral representation of the Poisson type for the modified Bessel-Clifford $E_\nu(x)$ [6].

Finally, from the generating function (2.6) written as a product of the two following hypergeometric functions

$${}_0F_{n-1}[1, 1, \ldots, 1; -\sqrt{x}\ t] \cdot {}_0F_{n-1}[1, 1, \ldots, 1; \sqrt{x}/t]$$

we can also infer another integral representation :

$$C^{(2n-1)}_{m,(n),m,0,(n-1),0}(x) = \frac{x^{-m/2}}{2\pi i} \cdot$$

$$\cdot \int_c^{(0^+)} t^{-61/m}\, C^{(n-1)}_{0,0,\ldots,0}(-\sqrt{x}\ t)\, dt \quad (m \in \mathbf{Z}) \tag{5.2}$$

wehre $\int_c^{(0^+)}$ stands for the Hankel contour integral.

This expression is closely related to the representation deduced by Agarwaf for the n-Bessel function $A_{m,n}(x)$ [1].

REFERENCES

[1] A.K. AGARWAL, A Generalization of Di-Bessel fucntion of Extorn, *Indian J.Pure Appl. Math.* 15 (1984), 139-148.

[2] R.P. AGARWAL, Sur une generalisation de la transformation de Hankel, *Ann. Soc. Sci. Bruxelles Ser. I* 64 (1950), 164-168.

[3] P.DELERRUE, Sur le Calcul Symbolique a n variables et les functions hyperbesseliennes, *Ann. Soc. Sci. Bruxelles Ser. I* 67 (1953), 83-104.

[4] P. DELERUE, Sur le Calcul Symbolique a *n* variables et les functions hyperbesseliennes, *Ann. Soc. Sci. Bruxelles Ser. I* 67 (1953), 229-274.

[5] I.H. DIMOVSKI and V.S. KIRYAKOVA, Generalized Poisson transmutations and corresponding representations of Hyper-Bessel functions, C.R. Acad. Bulgare Sci. 39 (1986), no.10, 29-32.

[6] N. HAYEK, Estudio de la ecuacion diferencial x $y' + (\nu + 1) y' + y = 0, y$ de sus aplicaciones, Collect. Math. 18 (1967), 57-174.

[7] N. HAYEK, Funciones de Bessel-Clifford de tercer orden, *Actas XII Jornadas Luso-Espanolas Braga* (1987), 346-351.

[8] N. HAYEK and V. HERNANDEZ-SUAREZ, Sobre las funciones de Bessel-Clifford de tercer orden, Actas XII C.E.D.Y.A. (II Congreso de Matematica Aplicada), Oviedo (1991).

[9] N. HAYEK and V. HERNANDEZ-SUAREZ, Representaciones integrales de las funciones de Bessel-Clifford de tercer orden, *Rev. Acad. Cien. Zaragoza* (1992) (To be published).

[10] P. HUMBERT, Les fonctions de Bessel du troisieme ordre, *Atti Pont. Accad. Sc. Nuovi Lincei* 83 (1930), 128-146.

[11] P. HUMBERT, Nouvelles remarques sur les functions de Bessel du troisieme ordre, *Atti Pont. Accad. Sc. Nuovi Lincei* 87 (1934), 323-331.

[12] P. HUMBERT et P. DELERUE, Sur l'equation differentielle de la funtion de Bessel du troisieme ordre et d'indices nuls, *Ann. Soc. Sci. Bruxelles Ser. I* 64 (1950), 159-163.

[13] V. KIRYAKOVA, An application of Meijer's G-function to Bessel-type operators, Constructive Function Theory, Varna'1984 (*Bulg. Acad. Sci.), Sofia* (1984), 457-462.

[14] V. KIRYAKOVA and S. SPIROVA, Representations of the solutions of Hyper-Bessel differential equations via Meijer's G- function, *Complex Analysis and Applications'* 87, Sofia (1989), 284-297.

[15] R.S. VARMA, Sur les fonctions de Bessel du troisieme ordre, *J. Ecole Polytech.* (III) 145 (1939), 33-35.

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(Decided to Professor S.A. Naimpally on his 62th Birthday)

TAYLOR INSTABILITY IN THE PRESENCE OF VARIABLE MAGNETIC FIELD AND SUSPENDED PARTICLES IN POROUS MEDIA

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ABSTRACT

Rayleigh-Taylor instability in the presence of vairable magnetic field and suspended particles through porous medium has been studied. The system is stable for stable density stratification. The magnetic field can stabilize a system which was unstable in its absence, for unstable density stratification. The medium permeability and suspended particles number density have decreasing as well as increasing effect on the growth rates.

1. INTRODUCTION

A comprehensive account of Rayleigh-Taylor instability under varying assumptions of hydromagnetics has been given by Chandrasekhar [2]. A contradiction was observed by Chandra [1], when he compared the theory for onset of convection in fluid heated from below and his experiment. He performed the experiement in an air layer and found that the instability depended on the depth of the layer. Scanlon and Segal [4] have studied the effect of suspended particles on the onset of Benard convection and found that critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma [6]. The effects of suspended particles and medium permeability were found to destabilize the layer. Sharma et al. [5] have considered the effect of suspended particles on the onset of bénard convection in hydromagnetics and found that suspended particles destabilize the layer whereas the effect of magnetic field was stabilizing.

Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases which in the process of their journey changes from solid to gas and vice-versa. The physical properties of comets, meteorites and interplanetary dust strongly suggest importance of porosity in astrophysical context) McDonnel [3]). The problem of the hydromagnetic stability of conducting fluid of variable density in the presence of suspended particles through porous medium plays an important role in stability of stellar atmospheres, in ground water hydrology or in oil recovery etc. In the present paper, a study has been made of the Rayleigh-Taylor instability of conducting fluid in the presence of suspended particles and variable magnetic field through porous medium with continuous stratificaions in fluid density, fluid viscosity, suspended particles number density and magnetic field.

2. PERTURBATION EQUATIONS

Consider an incompressible infinitely conducting fluid mixed with suspended particles through porous medium in which fluid density ρ, fluid viscosity μ and suspended particles number density N are functions of vertical component z only and layers of fluid are arranged in horizontal strata. A vaiable magnetic field $\vec{H}\,(H_0(z), 0, 0)$ and $\vec{g}\,(0, 0, -g)$ pervade the system. Initially the system is in a static state and the character of equilibrium of this static state is determined by supposing that the system is slightly disturbed and then following its further evolution.

Let $\rho, \mu, p, \vec{v}\,(u, v, w)$ denote the density, the viscosity, the pressure and the veloctiy of pure fluid respectively. $\vec{v}_p\,(l, r, s)$ and $N\,(\bar{x}, t)$ denote the velocity and the number density of suspended particles. $K = 6\pi/\mu\epsilon'$ is Stokes' drag, where ϵ' is the particle radius, μ_e is magnetic permeability and $\bar{x} = (x, y, z)$. Then the equations of motion and continuity for the fluid are

$$\frac{\rho}{\epsilon}\left[\frac{\partial \vec{v}}{\partial t}+\frac{1}{\epsilon}(\vec{v}\cdot\nabla)\,\vec{v}\right] = -\nabla p + \rho\vec{g} + \frac{KN}{\epsilon}(\vec{v}_p - \vec{v}) + \frac{\mu_e}{4\pi}(\nabla\times\vec{H})\times\vec{H} - \frac{\mu}{k_1}\vec{v}, \tag{1}$$

$$\nabla\cdot\vec{v} = 0, \tag{2}$$

where ϵ is porosity and k_1 is peremeability of the medium.

Since int he presence of suspended particles, the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude and opposite in sign in the equations of motion for suspended paticles. Assuming distances between paricles quite large compared with their diameter and so neglecting the buoyancy force exerted on particles and inter- particle reactions, the equaitons of motion and continuity for particles become

$$\frac{mN}{\epsilon}\left[\frac{\partial \vec{v}_p}{\partial t}+\frac{1}{\epsilon}\left(\vec{v}_p\cdot\nabla\right)\vec{v}_p\right] = mN\vec{g} + \frac{KN}{\epsilon}(\vec{v} - \vec{v}_p)\,. \tag{3}$$

$$\epsilon\,\frac{\partial N}{\partial t} + \nabla\cdot(N\vec{v}_p) = 0, \tag{4}$$

where mN is the mass of particles per unit volume.

Let $\delta\rho$, δp, $\vec{v}(u, v, w)$, $\vec{V}_p(l, r, s)$ and $\vec{h}(h_x, h_y, h_z)$ denote the perturbations in density ρ, pressure p, fluid velocity (0,0,0), particles velocity (0,0,0) and magnetic field $\vec{H}$ $(H_0(z), 0, 0)$ respectively. Then the linearized perturbation equations governing the motion are

$$\frac{\rho}{\in}\frac{\partial \vec{v}}{\partial t} = -\nabla \delta p + \vec{g}\delta\rho - \frac{\mu}{k1}\vec{v} + \frac{KN}{\in}(\vec{v_p} - \vec{v}) + \frac{\mu_e}{4\pi}[(\nabla \times \vec{h}) \times \vec{H} + (\nabla \times \vec{H}) \times \vec{h}] \tag{5}$$

$$\nabla . \vec{v} = 0 , \tag{6}$$

$$\nabla . \vec{h} = 0 , \tag{7}$$

$$\in \frac{\partial \vec{h}}{\partial t} = (\vec{H} . \nabla)\,\vec{v} - (\vec{v} - (\vec{v} . \nabla)\,\vec{H} , \tag{8}$$

$$m\,\frac{\partial \vec{v}_p}{\partial t} = k\,(\vec{v} - \vec{v}_p) , \tag{9}$$

$$\in \frac{\partial}{\partial t}\delta\rho = -w\,\frac{d\rho}{dz} . \tag{10}$$

3. DISPERSION RELATION AND DISCUSSION

Analyzing the disturbances into normal modes, we seek the solutions whose dependence on space-time coordinates is of the form

$$f(z)\exp\,(ik_x x + ik_y y + nt) , \tag{11}$$

where $f(z)$ is some function of z, n is the growth rate and k_x, k_y $\left[k = (k_x^2 + Sk_y^2)^{1/2}\right]$ are horizontal wave numbers.

Eliminating $\vec{v}_p$ between eqs. (5), (9) and using (11), we get

$$\left[\frac{\rho n}{\in} + \frac{\mu}{k_1} + \frac{m\,N\,n}{\in\,(1 + mn/K)}\right]\vec{v} = -\nabla\,\delta p + \frac{\mu_e}{4\pi}\left[\hat{i}\,h_z\,DH_0 + \hat{j}\,H_0\,(i\,k_x\,h_y - ik_y\,h_x)\right.$$

$$\left. + \hat{k}\,H_0\,(i\,k_x h_z - Dh_x - h_x)\,\frac{DH_0}{H_0}\right] + \vec{g}\delta\rho . \tag{12}$$

Writing eqs. (12) and (8) in the component form, and using (10) (11); we obtain

$$\left[\frac{\rho n}{\in} + \frac{\mu}{k_1} + \frac{mNn}{\in\,(1 + mn/K)}\right]u = -i\,k_x\,\delta p + \frac{\mu_e}{4\pi}h_z\,DH_0 , \tag{13}$$

$$\left[\frac{\rho n}{\epsilon}+\frac{\mu}{k_1}+\frac{mNn}{\epsilon(1+mn/K)}\right]v = -ik_y\,\delta p+\frac{\mu_e H_0}{4\pi}\left(ik_x h_y - ik_y h_x\right), \qquad (14)$$

$$\left[\frac{\rho n}{\epsilon}+\frac{\mu}{k_1}+\frac{mNn}{\epsilon\,(1+mn/K)}\right]w = -D\,\delta p+\frac{\mu_e H_0}{4\pi}\left(ik_x h_z - Dh_x - h_x\frac{DH_0}{H_0}\right)+\frac{g(D_\rho)\,w}{\epsilon n},$$

(15)

$$\epsilon\, n\, h_x = ik_x H_0\, u - w\, DH_0\,, \qquad (16)$$

$$\epsilon\, n\, h_y = ik_x H_0\, v\,, \qquad (17)$$

$$\epsilon\, n\, h_z = ik_x H_0\, w\,. \qquad (18)$$

Eliminating h_x, h_y, h_z from eqs. (13)-(15) with the help of eqs. (16)- (18), we get

$$\left(\frac{n'\rho}{\epsilon}+\frac{\mu}{k_1}\right)u = -ik_x\,\delta p+\frac{ik_x\mu_e H_0}{4\pi\epsilon n}\left(DH_0\right)w\,, \qquad (19)$$

$$\left(\frac{n'\rho}{\epsilon}+\frac{\mu}{k_1}\right)v = -ik_y\,\delta p+\frac{ik_x\,\mu_e}{4\pi\epsilon n}\left(i\,k_x\, v - i\,k_y\, u\right)+\frac{\mu_e H_0}{4\pi\epsilon n}\,ik_y\left(D\,H_0\right)w\,,$$

0)

$$\left(\frac{n'\rho}{\epsilon}+\frac{\mu}{k_1}\right)w = -D\,\delta p+\frac{\mu_e H_0}{4\pi\epsilon n}\Big[-k_x^2 H_0\, w - D\,(ik_x H_0\, u - w\, D\, H_0)$$

$$-(i\,k_x H_0\, u - w\, DH_0)\,\frac{DH_0}{H_0}\;+\;\frac{g\,(D\rho)\,w}{\epsilon\, n}\,, \qquad (21)$$

where

$$n' = n\left[1+\frac{mN}{\rho\,(1+mn/K)}\right]. \qquad (22)$$

Multiplying eq. (19) by $-ik_x$, (20) by $-ik_y$; adding and using (6), we get

$$\left(\frac{n'\rho}{\epsilon}+\frac{\mu}{k_1}\right)Dw = -k^2\,\delta p+\frac{\mu_e H_0^2\, k_x\, k_y}{4\,\pi\,\epsilon\, n}\,\zeta+\frac{\mu_e H_0 k^2 (DH_0)}{4\,\pi\,\epsilon\, n}\,w\,, \qquad (23)$$

where

$$\zeta = ik_x v - ik_y u. \qquad (24)$$

Multiplying (19) by $-ik_y$ (20) by ik_x and adding, we obtain

$$\zeta\; =\; 0, \qquad (25)$$

which together with (6) yields

$$u = \frac{ik_x}{k^2}\,Dw. \qquad (26)$$

Eliminating δp between eqs. (21), m (23) and using (25), (26), after a little algebra, we obtain

$$\frac{1}{\epsilon}\left[D\,(n'\,\rho\,D\,w) - n'\,K^2\,\rho\,w\right] + \frac{gK^2\,(D\rho)\,w}{n\,\epsilon} + \frac{\mu}{k_1}(D^2 - K^2)\,w + \frac{(D\mu)\,(Dw)}{k_1}$$

$$+\frac{\mu_e\,K_x^2}{4\,\pi\,\epsilon\,n}\left[H_0^2\,(D^2 - k^2)\,w + D.(H_0^2)\,Dw\right] = 0\,. \qquad (27)$$

Assume the stratifications in density, viscosity, number density and magnetic field of the form

$$\rho = \rho_0\,e^{\beta z}\,,\ \ \mu = \mu_0\,e^{\beta z}\,, N = N_0\,e^{\beta z}\,, H_0^2 = H_1^2\,e^{\beta z}\,, \qquad (28)$$

where ρ_0, μ_0, N_0, H_1 and β are constants. Here we consider the case of two free boundaries at z=0 and z=d. Assuming $\beta\,d << 1$, i.e. the variation in density at two neighbouring points in velocity field is much less than the average density, has a negligible effect on the inertia of the fluid. The appropriate boundary conditions are

$$w = D^2 w = 0\ (29) \qquad \text{at}\ z = 0\ \text{ and } z = d\,. \qquad (29)$$

The solution of (27) staisfying (29) is given by

$$w = A\,\sin\frac{m'\,\pi\,z}{d}\,, \qquad (30)$$

where m' is an integer. Substituting (30) into (27), we obtain the dispersion relation

$$n^3 + \left[\frac{K}{m} + \frac{KN_0}{\rho_0} + \frac{\nu_0\,\epsilon}{k_1}\right]n^2 + \left[k_x^2\,V_A^2 + \frac{\nu_0\,\epsilon\,K}{m\,K_1} - \frac{gK^2\,\beta}{L}\right]n$$

$$+\frac{k_x^2\,V_A^2\,K}{m} - \frac{gk^2\,\beta\,K}{m\,L} = 0, \qquad (31)$$

where $\nu_0 = \frac{\mu_0}{\rho_0}$, $V_A = \left(\frac{\mu_e\,H_0^2}{4\pi\rho}\right)^{1/2} = \left(\frac{\mu_e\,H_1^2}{4\,\pi\,\rho_0}\right)^{1/2}$ and $\frac{N}{\rho}$ are constant everywhere and $L = \left(\frac{m'\,\pi}{d}\right)^2 + k^2$. For $\beta < 0$ (stable stratification), eq. (31) does not allow any positive root of n and so the system is stable. For $\beta > 0$ (unstable stratification), the system is stable or unstable according as

$$k_x^2 V_A^2 \gtrless \frac{gk^2\beta}{L}\,. \qquad (32)$$

In the absence of magnetic field, the system is unstable for $\beta > 0$. However, the system can be stabilized by magnetic field, as can be seen from eq. (32), if

$$V_A^2 > \frac{g\,\beta\,K^2}{k_x^2\,L}\,.$$

If $\beta > 0$ and $V_A^2 < \dfrac{gK^2\beta}{k_x^2\,L}$, eq. (31) has at least one positive root. Let this positive root be n_0, then

$$n_0^3 + \left[\frac{K}{m} + \frac{KN_0}{\rho_0} + \frac{\nu_0 \in}{k_1}\right] n_0^2 + \left[k_x^2\,V_A^2 + \frac{\nu_0 \in K}{mk_1} - \frac{gK^2\,\beta}{L}\right] n_0$$

$$+\frac{k^2{}_x\,V_A^2\,K}{m} - \frac{gk^2\beta\,K}{m\,L} = 0. \qquad (33)$$

To find the role of permeability and suspended particles number density on the growth rate unstable modes, we examine the natures of dn_0/dk_1 and dn_0/dN_0. Equation (33) yields

$$\frac{dn_0}{dk_1} = \frac{n_0\nu_0 \in \left(n_0 + \dfrac{K}{m}\right)}{k_1^2\left[3n_0^2 + 2\left(\dfrac{K}{m} + \dfrac{KN_0}{\rho_0} + \dfrac{\nu_0\in}{k_1}\right)n_0 + \left(k_x^2\,V_A^2 + \dfrac{\nu_0 \in K}{mk_1} - \dfrac{gK^2\beta}{L}\right)\right]}. \qquad (34)$$

This shows that growth rates decrease of increase with permeability according as the denominator is negative or positive.

Equation (33) also yields

$$\frac{dn_0}{dN_0} - \frac{\left(\dfrac{K}{\rho_0}\right)n_0^2}{3\,n_0^2 + 2\left(\dfrac{K}{m} + \dfrac{KN_0}{\rho_0} + \dfrac{\nu_0\in}{k_1}\right)n_0 + \left(k_x^2\,V_A^2 + \dfrac{\gamma_0 \in K}{mk_1} - \dfrac{gK^2\beta}{L}\right)}.$$

Thus, if in addition to $K_x^2 V_A^2 < \dfrac{gk^2\,\beta}{L}$ we have the condition

$$\left|K_x^2\,V_A^2 - \frac{gk^2\beta}{L}\right| \;<\; 3\,n_0^2 + 2\,n_0\left(\frac{K}{m} + \frac{KN_0}{\rho_0} + \frac{\nu_0 \in}{k_1}\right) + \frac{\nu_0 \in K}{m\,k_1}\,,$$

then dn_0/dN_0 is always negative and so the growth rate decreases with the increase in the suspended particles number density. However the growth rate increases with the number density for the region

$$\left| k_x^2 V_A^2 - \frac{gK^2\beta}{L} \right| > 3\, n_0^2 + 2n_0 \left(\frac{K}{m} + \frac{KN_0}{\rho_0} + \frac{\nu_0 \in}{k_1} \right) + \frac{\nu_0 \in K}{m\, k_1} \, .$$

REFERENCES

[1] K. Chandra, Instability of fluid heated from below, *Proc. Roy. Soc. London* Sect.A 164 (1938), 231.

[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability,* Dover Publication, New York, 1981.

[3] J.W. McDonnel, *Cosmic Dust,* John Wiley and Sons, Toronto, p. 330.

[4] J.W. Scanlon and L.A. Segel, Some effects of suspended particles on the onset of Bénard convection, *Phys. Fluids* **16** (1973), 1573-1578.

[5] R.C. Sharma, K. Prakash and S.N. Dube, Effect of suspended particles on the onset of bénard convections in hydromagnetics *Acta Phys. Hungar.* **40** (1976), 3-10.

[6] R.C. Sharma and K.N. Sharma, Thermal instability of fluids through a porous medium in the presence of suspended particles, rotation and solute gradient, *J. Math. Phys. Sci.* **16** (1982), 167-181.

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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

THE DISTRIBUTON OF THE QUOTIENT OF PRODUCTS OF RATIONAL POWERS OF INDEPENDENT RANDOM VARIABLES

By

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ABSTRACT

In this paper an attempt has been made to present the distribution of the quotient of products of rational powers of independent Stochastic varibales with probability denstiy function associated with Beta distribution involving Fox's H- fucntion.

1. Introduction

A number of statistical distribuions have been studied from time to time by several authors. For example, Methai and Saxena [1], and Srivastava and Singhal [3], studied a general class of distributions, whose probability density fucntion involves a hypergeometric funtion ${}_2F_1$ and Fox's H-function respectively. Further, Exton [2] and Agrawal [5] considered the family of distributions which have the probability density function in terms of the product of several generalized hypergeometric funcions ${}_pF_q$ and the H-function of several varibales respectively. In this paper we have considered the probability denstiy fucntion associated with Beta distribution involving Fox's H-function, to study the distribution of the quotient of products of rational powers of independent random variables.

Let x_i $(i = 1, ..., r)$ be r independent random variables such that their probability density fucntion, which follows from [5,p.93,eq.(2.1)] is given by

$$f_i(x_i) = \begin{cases} L_i x_i^{\lambda_i - 1}(1 - x_i)^{\gamma_i} H^{m_i, n_i}_{p_i, q_i}\left[a_i x_i^{u_i}(1 - x_i)^{v_i} \middle| \begin{matrix} (a_j^{(i)}, \alpha_j^{(i)})_{1,p_i} \\ (b_j^{(i)}, \beta_j^{(i)} 1)_{1,q_i} \end{matrix} \right], \\ 0, \quad \text{elsewhere} \end{cases}$$

$$0 \le x_i \le 1, i = 1, ..., r \qquad \text{... (1.1)}$$

where

$$L_i^{-1} = H^{m_i, n_i + 2}_{p_i + 2, q_i + 1}\left[\begin{matrix} (1 - \lambda_i, u_i), (-\gamma_i, v_i), (a_j^{(i)}, \alpha_j^{(i)})_{1,p_i} \\ (b_j^{(i)}, \beta_j^{(i)})_{1,q_i}, (-\lambda_i - \gamma_i, (-\lambda_i - \gamma_i, u_i + v_i) \end{matrix} \right] \qquad \text{... (1.2)}$$

$H^{m,n}_{p,q}$ [x] denotes the well-known Fox's H-function [4] and the following conditions are satisfied

(i) $u_i, v_i \geq 0$ (not both zero simultaneously)

(ii) $\lambda_i + u_i \min\limits_{1 \leq j \leq m_i} \left(b_j^{(i)} / \beta_j^{(i)} \right) > 0$

(iii) $\gamma_i + v_i \min\limits_{1 \leq j \leq m_i} \left(b_j^{(i)} / B_j^{(i)} \right) + 1 > 0$

(iv) $A = \sum\limits_{j=1}^{m_i} \beta_j^{(i)} - \sum\limits_{j=m_i+1}^{q_i} \beta_j^{(i)} + \sum\limits_{j=1}^{n_i} \alpha_j^{(i)} - \sum\limits_{j=n_i+1}^{p_i} \alpha_j^{(i)} > 0$

(v) The parameters involved in (1.1) are so restricted that $f_i(x_i)$ remains positive for $0 \leq x_i \leq 1$, $(i = 1, \ldots, r)$ ... (1.3)

2. Distribution of the quotient of products of rational powers of several random variables

The theorem given below gives us the distribution of the quotient of products of rational powers of several independent random variables associated with the probability denstiy function defined by (1.1).

Theorem : Let x_i $(i = 1, \ldots, r)$ be r independent random variables, where x_i has the probabilty density function defined by (1.1). Then the probability denstiy function $h(u)$ of

$$U = \frac{x_1^{p_1} \ldots x_n^{p_n}}{x_{n+1}^{p_{n+1}} \ldots x_r^{P_r}} \; (1 \leq n \leq r) \text{ with } p_i \geq o \; (i = 1, \ldots, r) \qquad \ldots (2.1)$$

is given by

$$h(u) = \prod_{i=1}^{r} \left\{ L_i \sum_{j=1}^{m_i} \sum_{\mu_i=0}^{\infty} \psi_\iota (m_i, \mu_i, \rho_{ij}) \frac{(-1)^{\mu_i} a_i^{ij}}{\mu_i! \, \beta_j^{(i)}} \Gamma(1 + \gamma_i + v_i \rho_{ij}) \right\} .$$

$$. \; H^{n, r-n}_{r,r} \left[U \left| \begin{matrix} (1 - \lambda_i - u_i \rho_{ij}, p_i)_{n+1, r} \, (1 + \gamma_i + \lambda_i + u_i \rho_{ij} + v_i \rho_{ij}, p_i)_{1,n} \\ (\lambda_i + u_i \rho_{ij}, p_i)_{1,n} \, (-\gamma_i - \lambda_i - \mu_i \rho_{ij} - v_i \rho_{ij}, p_i)_{n+1, r} \end{matrix} \right. \right] \qquad (2.2)$$

where

$$\rho_{ij} = \frac{b_j^{(i)} + \mu_i}{\beta_j^{(i)}} \text{ and } \psi_i (m_i, \mu_i, \rho_{ij}) =$$

$$\frac{\prod\limits_{\substack{h=1\\h\neq j}}^{m_i} \Gamma\left(b_n^{(i)} - \beta_h^{(i)}\, \rho_{ij}\right) \prod\limits_{h=1}^{n_i} \Gamma\left(1 - a_h^{(i)} + \alpha_h^{(i)}\, \rho_{ij}\right)}{\prod\limits_{h=m_i+1}^{q_i} \Gamma\left(1 - b_h^{(i)} + \beta_h^{(i)}\, \rho_{ij}\right) \underset{h=n_i+1}{p_i} \Gamma\left(a_h^{(i)} - \alpha_h^{(i)}\, \rho_{ij}\right)} \qquad \text{... (2.3)}$$

The theorem is valid provided that $\left\{\lambda_i \pm sP_i + u_i \min\limits_{1 \le j \le m_i} \left(b_j^{(i)}/\beta_j^{(i)}\right)\right\} > 0$ and the conditions (*i*), (*iii*), (*iv*) and (*v*) of (1.3) are satisfied.

proof : Let

$$V = \log U = \sum_{i=1}^{n} \log x_i^{P_i} - \sum_{i=n+1}^{r} \log x_i^{P_i} = \sum_{i=1}^{n} \log Y_i - \sum_{i=r+1}^{r} \log Y_i.$$

We first obtain the probability function of $Y_i = X_i^{P_i}$. For this we make use of a known theorem [7, p. 70, eg. (2)], with (1.1) and easily get the p.d.f. $G_i\,(Y_i)$ of Y_i as follows :

$$G_i\,(Y_i) = \begin{cases} L_i P_i^{-1}\, Y_i^{\lambda_i/P_i - 1} \left(1 - Y_i^{1/P_i}\right)^{\gamma_i} H_{p_i, q_i}^{m_i, n_i}\left[a_i\, Y_i^{u_i/P_i} \left(1 - Y_i^{1/P_i}\right)^{v_i} \,\middle|\, \begin{matrix} \left(a_j^{(i)}, \alpha_j^{(i)}\right)_{1, p_i} \\ \left(b_j^{(i)}, \beta_j^{(i)}\right)_{1, q_i} \end{matrix}\right], & 0 \le Y_i \le 1 \\ 0, \text{ elsewhere} \end{cases}$$

$$i = 1, \ldots, r \qquad \text{... (2.4)}$$

Next, to obtain the probability density fucntion of U, we use the method of Laplace transform and its inverse.

Let $\phi_v(s)$ denote the Laplace transform of V, then

$$\phi_v^{(s)} = E\,(e^{-sv}) = E\left(\prod_{i=1}^{n} Y_1^{-s} \prod_{i=n+1}^{r} Y_i^{s}\right)$$

$$= \prod_{i=1}^{n}\left[\int_0^{\infty} G_i\,(y_i)\, Y_i^{-s}\, dy_i\right] \prod_{i=n+1}^{r}\left[\int_0^{\infty} G_i\,(Y_i)\, Y_i^{S}\, dY_i\right] \qquad \text{... (2.5)}$$

Substituting the value of $G_i\,(Y_i)$ from equaion (2.4) in (2.5), changing the variable of integration slightly and evaluating the integrals thus obtained with the help of [4,p.61,eq.(5.2.3)], we get

$$\phi_v(s) = \prod_{i=1}^{n}\left\{L_i\, H_{p_i+2,\, q_i+1}^{m_i,\, n_i+2}\left[a_i \,\middle|\, \begin{matrix} (1 - \lambda_i + sp_i\,,\, u_i)\,,\, (-\gamma_i\,,\, v_i)\,,\, (a_j^{(i)}\,,\, \alpha_j^{(i)})_{1, p_i} \\ (b_j^{(i)}\,,\, \beta_j^{(i)})_{1, q_i}\, (-\lambda_i + sp_i - \gamma_i\,,\, u_i + v_i) \end{matrix}\right]\right\},$$

$$\prod_{i=n+1}^{r}\left\{L_i\, H_{p_i+2,\, q_i+1}^{m_i,\, n_i+2}\left[\, a_i \left|\begin{matrix}(1-\lambda_i - sp_i\,,\, u_i)\,,\,(-\gamma_i\,,\, v_i)\,,\,(a_j^{(i)}\,,\,\alpha_j^{(i)})_{1,\,p_i}\\ (b_j^{(i)}\,,\,\beta_j^{(i)})_{1\,,\,q_i}\,(-\lambda_i - sp_i - \gamma_i\,,\, u_i + v_i)\end{matrix}\right.\right]\right\},$$

... (2.6)

Now we express each *H*-function invloved in (2.6) into series form [4, p. 12, eq. (2.2.4)] and collect the terms containing s, viz.

$$\prod_{i=1}^{n}\left\{\frac{\Gamma(\lambda_i - sp_i + u_i\,\rho_{ij})}{\Gamma(1+\gamma_i+\lambda_i+u_i\,\rho_{ij}+v_i\,\rho_{ij}-sp_i)}\right\}\prod_{i=n+1}^{r}\left\{\frac{\Gamma(\lambda_i + sp_i + u_i\,\rho_{ij})}{\Gamma(1+\gamma_i+\lambda_i+u_i\,\rho_{ij}+v_i\,\rho_{ij}+sp_i)}\right\}$$

... (2.7)

On taking the inverse Laplace transform of the expression (2.7) and substituting the value thus obtained in the expanded series of (2.6), we shall arrive at the desired result (2.2) after a little simplification.

3. Particular Cases

The probability density function given by (1.1) is quite general in nature due to the presence of Fox's H-function and a number of known distributions can be derived as specialized or limiting cases of this distribution. For example, if we take $\alpha_j^{(i)} = \beta_j^{(i)} = 1, m_i = 1, n_i = p_i$ and $q_i \to q_i + 1,\ a_j^{(i)} \to 1 - a_j^{(i)}\,,\ b_j^{(i)}\,, \to 1 - b_j^{(i)}$ in (1.1), (1.2) and (2.2), we arrive at the following :

Corollary 1 : If

$$f_i\,(x_i) \;=\; \begin{cases} M_i\, x_i^{\lambda_i - 1}\;(1-x_i)^{\gamma_i}\;{}_{p_i}F_{q_i}\left[a_j^{(i)}\,;\, b_j^{(i)}\,;\, -\,a_i x_i^{u_i}\,(1-x_i)^{v_i}\right], & 0 \le x_i \le 1\\ 0, & \text{elsewhere}\end{cases}$$

$$i = 1,\, \ldots,\, r$$

... (3.1)

then the density function of U as defined by (2.1) is given by

$$h_1^{(u)} = \prod_{i=1}^{r}\left\{M_i \sum_{\mu_i=0}^{\infty}\left[\prod_{j=1}^{p_i}\Gamma\left(a_j^{(i)}+\mu_i\right)\right]\left[\prod_{j=1}^{q_i}\Gamma\,(b_j^{(i)}+\mu_i)\right]^{-1}\right.,$$

$$\times\frac{(-a_i)^{\mu_i}}{\mu_i\,!}\;\Gamma\,(1+\gamma_i+v_i\,\mu_i)\;\; H_{r,r}^{n,\,r-n}\,[\,U\,]$$

$$\begin{matrix}(1-\lambda_i - u_i\,\mu_i\,,\,p_i\,)_{n+1,\,r}\,(1+\gamma_i+\lambda_i+u_i\,\mu_i+v_i\,\mu_i\,,\,p_i)_{1,\,n}\\ (\,\lambda_i+\mu_i\;\;\mu_i\,,\,P_i\,)_{1,\,n}\,(-\,\gamma_i-\lambda_i-u_i\,\mu_i-v_i\,\mu_i\,,\,p_i)_{n+1,\,r}\end{matrix}$$

... (3.2)

where

$$M_i^{-1} = H_{p_i+2,\,q_i+2}^{1,\,p_i+2}\left[\,a_i\left|\begin{matrix}(1-\lambda_i\,,\,u_i)\,,\,(-\gamma_i\,,\,v_i)\,,\,(1-a_j^{(i)}\,,\,1)_{1,\,p_i}\\ (0,\,1)\,,\,(1-b_j^{(i)}\,,\,1)_{1,\,q_i}\,(-\,\lambda_i-\gamma_i\,,\,u_i+v_i)\end{matrix}\right.\right] \quad \ldots (3.3)$$

and the conditions easily obtainable from those stated with the main theorem are satisfied

On taking $\gamma_i = 0$ and $v_i = 0$ in (1.1) and (2.2), we arrive at the following :

Corollary 2 : If

$$f_i(x_i) = \begin{cases} K_i \, x_i^{\lambda_i - 1} \, H_{p_i, q_i}^{m_i, n_i}\left[a_i \, x_i^{u_i} \left| \begin{matrix} (a_j^{(i)}, \alpha_j^{(i)})_{1, p_i} \\ (b_j^{(i)}, \beta_j^{(i)})_{1, q_i} \end{matrix} \right. \right], \; 0 \leq x_i \leq 1 \\ 0, \quad \text{elsewhere} \end{cases}$$

$$i = 1, \ldots, r \qquad \ldots (3.4)$$

then the density function of $\underset{\cdot}{U}$ is given by

$$h_2(u) = \prod_{i=1}^{r} \left\{ K_i \sum_{j=1}^{m_i} \sum_{\mu_i = 0}^{\infty} \psi_\iota \; (m_i, \mu_i, \rho_{ij}) \; \frac{(-1)^{\mu_i} \, a_i^{\rho_{ij}}}{\mu_i ! \; B_j^{(i)}} \right\},$$

$$\cdot \; H_{r, r}^{n, r-n}\left[u \left| \begin{matrix} (1 - \lambda_i - u_i \, \rho_{ij}, p_i)_{n+1, r} \, (1 + \lambda_i + u_i \, \rho_{ij}, p_i)_{1, n} \\ (\lambda_i + u_i \, \rho_{ij}, p_i)_{1, n} \, (-\lambda_i - u_i \, \rho_{ij}, p_i)_{n+1, r} \end{matrix} \right. \right] \qquad \ldots (3.5)$$

where

$$K_i^{-1} = H_{p_i + 1, q_i + 1}^{m_i, n_i + 1}\left[a_i \left| \begin{matrix} (1 - \lambda_i, u_i), (a_j^{(i)}, \alpha_j^{(i)})_{1, P_i} \\ (b_j^{(i)}, \beta_j^{(i)})_{1, q_i}, (-\lambda_i, u_i) \end{matrix} \right. \right] \qquad \ldots (3.6)$$

and the conditions easily obtainable from those of the main theorem are satisfied.

(iii) On putting $p_i = 1$ $(i = 1, \ldots, r)$ and $X_2 \ldots X_n = 1 = X_{n+2} \ldots X_r$ in (2.2) we obtain a result essentially similar to the one variable analogue of the result given by Goyal and Audich [6,p.9,eq.(2.12)].

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REFERENCES

[1] A.M.Mathai and R.K.Saxena, On a generalized hypergeometric distribution, *Metrika* **11** (1966), 127-131.

[2] H.Exton, *Handbook of Hypergeometric Integrals,* Ellis Horwood Ltd., Chichester, 1978, p.182.

[3] H.M.Srivastava and J.P. Singhal, On a class of generalized hypergeometric distributions, *Jñānābha sect.A* **2** (1972), 1-9.

[4] H.M. Srivastava, K.C. Gupat and S.P. Goyal, *The H-Functions of One and Two Variables with Applications,* South Asian Publishers, New Delhi, 1982.

[5] R.K. Agrawal, A general class of univariate distributions involving the H-function of several varibales, *Jnanabha* **9** (1980), 89-97.

[6] S.P. Goyal and Sunil Audich, The distribution of the ratio of two random vairbales associated with Fox's H-function, *Ganita Sandesh* **4** (1990), 6-11.

[7] V.K. Rohatgi, *An Introduction to Probability Theory and Mathematical Statistics,* Wiley Eastern Limited, New Delhi, 1975.

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(Dedicated to Professor S.A.Naimpally on his 62the Birthday)

INTEGRATION OF CERTAIN PRODUCTS ASSOCIATED WITH THE MULTIVARIABLE *H*-FUNCTION AND A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

In the present paper we evaluate three finite integrals innvolving the products of Fox's H-function, the multivariable H-function and a general class of polynomials. On account of the most general nature of the functions invloved herein a very large number of known and new integrals involving simpler special functions and orthogonal polynomials follow as particular cases of our main integrals.

1. INTRODUCTION

Srivastava [5] introduced the general class of polynomials (see also Srivastava and Singh [9]) :

$$S_{\lambda}^{w}[x] = \sum_{\alpha=0}^{[\lambda/w]} \frac{(-\lambda)_{w\alpha}}{\alpha!} L_{\lambda,\alpha} x^{\alpha}, \ \lambda = 0, 1, 2, \ldots, \qquad \ldots(1.1)$$

where w is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}$ $(\lambda, \alpha \geq 0)$ are arbitrary constants real or complex. By suitably specializing the coefficeints $L_{\lambda,\alpha}$, the general class of polynomials can be reduced to a large spectrum of polynomials as cited in the papers referred to above.

The series representaion of Fox's H-function (see [1] and [4])

$$H_{p,q}^{m,n}\left[\left|\begin{matrix}(e_p, E_p)\\(f_g, F_g)\end{matrix}\right.\right] = \sum_{g=1}^{m}\sum_{s=0}^{\infty} (-1)^s \, \phi(\eta_s) \left\{s!\, F_g\right\}^{-1}, \qquad \ldots(1.2)$$

where

$$\phi(\eta_s) = \prod_{j=1, j\neq g} \Gamma(f_j - F_j\eta_s) \prod_{j=1}^{n} \Gamma(1 - e_j + E_j\eta_s)$$

$$\cdot\left\{\prod_{j=m+1}^{q} \Gamma(1 - f_j - F_j\eta_s) \prod_{j=n+1}^{p} \Gamma(e_j - E_j\eta_s)\right\}^{-1}$$

and $\eta_s = (f_g + s)/F_g$ ... (1.3)

For the multivariable $\bar{H}$-function defined by Srivastava and Panda ([6] and [7]; see also [8]), we derive three finite integrals in the next section.

2. THE MAIN INTEGRALS

The following integrals have been established in this section :

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\{hx+k(1-x)\}^{-2\rho}$$

$$\qquad {}_2F_1\left[u, v; (u+v+1)/2; hx\,\{hx+k(1-x)\}^{-1}\right]$$

$$\cdot\ H^{m,n}_{p,q}\left[z\,\{hkx(1-x)\}^{\sigma}\,\{hx+k(1-x)\}^{-2\sigma}\ \middle|\ \begin{matrix}(e_p, E_p)\\ (f_q, F_q)\end{matrix}\right]$$

$$\cdot\ S^{w}_{\lambda}\left[z\,\{hkx(1-x)\}^{\mu}\,\{hx+k(1-x)\}^{-2\mu}\right]$$

$$H^{0,B:(M',N');\ldots;(M^{(r)},N^{(r)}),}_{A,C:(P',Q');\ldots;(p^{(r)},Q^{(r)}),}\left(\frac{z_1\{hkx(1-x)\}^{\sigma_1}}{\{hx+k(1-x)\}^{2\sigma_1}},\ldots,\frac{z_r\{hkx(1-x)\}^{\sigma_r}}{\{hx+k(1-x)\}^{2\sigma_r}}\right. dx\mathrm{w}$$

here

$$= \sum_{\alpha=0}^{[\lambda/w]}\ \sum_{g=1}^{m}\ (-1)^s\,\phi(\eta_s)\,z^{\eta_s}\{s\,!\,F_g\}^{-1}\,\frac{(-\lambda)_{w\alpha}}{\alpha\,!}\,L_{\lambda,\alpha}$$

$$\cdot\ 2^{1-2p-2\sigma-\eta_s-2\mu\alpha}\,\pi\,\Gamma\{(u+v+1)/2\}\{(hk)^{\rho}\,\Gamma\{(u+1)/2\}\,\Gamma\{(v+1)/2\}\}^{-1}$$

$$\cdot\ H^{0,B+2:(M',N');\ldots,(m^{(r)},N^{(r)})}_{(A+2,C+2:(P',Q');\ldots;(p^{(r)},Q^{(r)})}\left(\begin{matrix}[1-\rho-\sigma\eta_s-\mu\alpha:\sigma_1,\ldots,\sigma_r],\\ [(c):\psi',\ldots,x^{(r)}],\end{matrix}\right.$$

$$[-\sigma-\sigma\eta_s-\mu\alpha+(u+v+1)/2:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]:$$

$$[-\rho-\sigma\eta_s-\mu\alpha+(u+1)/2:\sigma_1,\ldots,\sigma_r],\ [-\rho-\sigma\eta_s-\mu\alpha+(v+1)/2:\sigma_1,\ldots,\sigma_r],$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots,[(b^{(r)}):\phi^{(r)}];\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1 4^{-\sigma_1},\ldots,z_r 4^{-\sigma_r}\right) \qquad \ldots (2.1)$$

where $\sigma > 0, \mu > 0, \sigma_i > 0,\ Re\ \{(1-u-v)/2\} > 0$,

$Re\ (\rho + \sigma f_l/F_l + \sum_{i=1}^{r}\sigma_i\ a_j^{(i)}/\delta_j^{(i)}) > 0,\ l = 1, \ldots, m;\ j = 1, \ldots, M^{(i)}$,

$T > 0,\ |\arg z| < T\pi/2,\ |\arg z_i)| < T_i\ \pi/2,\ h, k$ are non-zero constant and $\{hx + k(1-x)\}$ is non-zero where $0 \le x \le 1$, w is an artitrary positive integer and the coefficents $L_{\lambda,\alpha}\ (\lambda, \alpha \ge 0)$ an arbitrary constants real or complex

$$\left(T = \sum_{1}^{n} E_i - \sum_{n+1}^{p} E_i + \sum_{1}^{m} F_i - \sum_{m+1}^{q} F_i\right).$$

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-v}\left\{hx+k(1-x)\right\}^{-2\rho+-1}$$

$$\cdot\ {}_2F_1\left[u-1-u : v{:}hx\left\{hx+k(1-x)\right\}^{-1}\right]$$

$$\cdot\ H_{p,q}^{m,n}\left[z\left\{hkx(1-x)\right\}^{\sigma}\left\{hx+k(1-x)\right\}^{-2\sigma}\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right.\right]$$

$$\cdot\ S_\lambda^w\left[\left\{hkx(1-x)\right\}^{\mu}\left\{hx+k(1-x)\right\}^{-2\mu}\right]$$

$$\cdot\ H_{A,C:(P',Q');\ldots,(P^{(r)},Q^{(r)})}^{0,B:(M',N');\ldots,(M^{(r)},N^{(r)})}\left(\frac{z_1\left\{hkx+(1-x)\right\}^{\sigma_1}}{\left\{hx+k(1-x)\right\}^{2\sigma_1}},\ldots,\frac{z_r\left\{hkx+(1-x)\right\}^{\sigma_r}}{\left\{hx+k(1-x)\right\}^{2\sigma_r}}\right)dx$$

$$=\sum_{\alpha=0}^{[\lambda/w]}\sum_{g=1}^{m}\sum_{s=0}^{\infty}(-1)^s\phi(\eta_s)z^{\eta_s}\left\{s!F_g\right\}^{-1}\frac{(-\lambda)_{w\alpha}}{\alpha!}L_{\lambda,\alpha}$$

$$\cdot\ (hk)^{\sigma\eta_s+\mu\alpha}2^{1-2\sigma-2\rho\eta_s-2\mu\alpha}\Gamma(v)\Pi\left\{\Gamma\{(u+v)/2\}\Gamma\{(1-u+v)/2\}\right\}^{-1}$$

$$H_{A+2,C+2:(P',Q');\quad;Q^{(r)},P^{(r)}}^{0,\quad B+2:(M',N');\ldots;(M^{(r)},N^{(r)})}\left(\begin{matrix}[1-\rho-\sigma\eta_s-\mu\alpha:\sigma_1,\ldots,\sigma_r],\\ [(c):\psi',\ldots,\psi^{(r)}],\end{matrix}\right.$$

$$[v-\rho-\sigma\eta_s-\mu\alpha:\sigma_1,\ldots,\sigma_r],[(a):\theta',\ldots,\theta^{(r)}]:$$

$$[-\rho-\sigma\eta_s-\mu\alpha+(1-u+v)/2:\sigma_1,\ldots,\sigma_r],[1-\rho-\sigma\eta_s-\mu\alpha+(u+v)/2:\sigma_1,\ldots,\sigma_r],[(b')$$

$$[(d'):\delta'];\ldots,[(d^{(r)});\delta^{(r)}];\quad z_1(hk)^{\sigma_{1/4}\sigma_1},\ldots,z_r(hk)^{\sigma_{r/4}\sigma_r}\qquad\ldots(2,2)$$

where $\sigma > o$, $\mu > 0$, $\sigma_i > 0$, $\mathrm{Re}(v) > 0$,

$\mathrm{Re}\,(\rho+\sigma f_l/F_l+\sum_{i=1}^{r}\sigma_r\ a_j^{(i)}/deltd_j^{(i)}) > 0,\ l=1,\ldots,m;\ j=1,\ldots,M^{(i)}$,

$|\arg z| < T\pi/2$, $|\arg(z_i)| < T_i\pi/2$, $T>0$, $T_i>0$ and h,k are are non-zero constants and $\{hx+k(1-x)\}$ is non-zero where $0 \le x \le 1$. w is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda,\alpha \ge 0)$ are arbitrary constants real or complex.

$$\cdot\int_0^1 t^{u-1}(1-t)^{a-2u}(1+vt)^{u-a-1}\ {}_2F_1\left[a,b;1+a-b;(1+v)t\left\{(1+vt)\right\}\right]$$

$$\cdot\ H_{p,q}^{m,n}\left[z\left\{\frac{4(1+v)(t+vt^2)}{(1-t)^2}\right\}^{h}\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right|\right]\ S_\lambda^w\left[\left\{\frac{4(1+v)(t+vt^2)}{(1-t^2)}\right\}^{\mu}\right]$$

$$. \; H_{A, C: (p', Q'); \ldots;)P^{(r)}, Q^{(r)})}^{0, 0: (M', N'); \ldots; (M^{(r)}, N^{(r)})} \left(z_1, \ldots, z_i \left\{ \frac{4(1+v)(t+vt^2)}{(1-t)^2} \right\}^{k_i}, \ldots, z_r \right) dt$$

$$= \sum_{\alpha=0}^{[\lambda/w]} \sum_{g=1}^{m} \sum_{s=0}^{\infty} (-1)^s \phi(\eta_s) z^{\eta_s} \left\{ s! \, F_g \right\}^{-1} \frac{(-\lambda)_{w\alpha}}{\alpha!} L_{\lambda, \alpha}$$

$$. \; 2^{a-2u} (1+v)^{-u} \Gamma(1+a/2) \Gamma(1+a-b) \left\{ \sqrt{\pi} \; \Gamma(1+a) \Gamma(1-b+a/2) \right\}^{-1}$$

$$. \; H_{A, C: (P', Q'); \ldots; (P^{(i)}+2, Q^{(i)}+2); \ldots; (P^{(r)}, Q^{(r)})}^{0, 0: (M', N'); \ldots; (m^{(i)}+2, N^{(i)}+1; \ldots; (M^{(r)}, N^{(r)})} \left(\begin{matrix} [(a): e', \ldots, e^{(r)}] : \\ [(c): \psi', \ldots, \psi^{(r)}] : \end{matrix} \right.$$

$$[(b'): \phi']; \ldots; [1-u-h\eta_s-\mu\alpha : k], [(b^{(i)}): \phi^{(i)}],$$

$$[(d'): \delta']; \ldots; [-u-h\eta_s-\mu\alpha+(1+a)/2 : k], [1-u-h\eta_s-\mu\alpha+\frac{a}{2}-b : k],$$

$$\left. \begin{matrix} [1+a-b-u-h\eta_s-\mu\alpha : k]; \ldots; [(b^{(r)}): \phi^{(r)}]; \\ [(d^{(i)}): \delta^{(i)}] \qquad\qquad ; \ldots; [(d^{(r)}): \delta^{(r)}]; \end{matrix} \; z_1, \ldots, z_r \right), \qquad \ldots (23)$$

where $v > -1, h > 0, k > 0, \mu > 0, \; \mathrm{Re}(a-2u-2b+2) > 0,$

$Re(u + h f_l/F_l + k d_j^{(i)}/\delta_j^{(i)}) > 0, \quad l = 1, \ldots, m; \; j = 1, \ldots, M^{(i)},$

$Re\left(a - 2u - 2h(e_{l'}-1)/E_{l'} - 2k(1 - b_j^{(i)}/\phi_j^{(i)}) + 1\right) > 0, \; l' = 1, \ldots, n;$

$j = 1, \ldots, N^{(i)}, \; |\arg z| < T\pi/2, \; |\arg(z_i)| < T_i \pi/2, \; T > 0,$

$T_i > 0, i = 1, \ldots, r$ and w is an arbitrary positive integer and the coefficients $L_{\lambda, \alpha}$ $(\lambda, \alpha \geq 0)$ are arbitrary constants real of complex.

Proof : To establish (2.1), express the general class of polynomials occurring in the integrand of (2.1) in series form given by (1.1) and then interchange the order of summation and integration (which is permissible under the conditions stated above), evaluate the inner integral with the help of a result of Chaurasia ([2],p.67,eq.(2.1)), we arrive at the desired result.

The proofs of the integrals (2.2) and (2.3) can be developed by proceeding on similar lines with the help of known integrals ([2],p.67,eq.(2.2) and p.68,eq.(2.3)).

3. SPECIAL CASES

(i) Giving suitable values to the parameters and using a result ([6],p.138), the results (2.1) to (2.3) can be reduced to the following integral formulae involving the Lauricella's fucntions :

$$\int_0^1 x^{\rho-1} (1-x)^{\rho-1} \left\{ hx + k(1-x) \right\}^{-2\rho} {}_2F_1\left[u, v; (u+v+1)/2; \frac{hx}{\{hx + k(1-x)\}} \right]$$

$$. H^{m,n}_{p,q}\left[\frac{z\{hkx(1-x)\}^{\sigma}}{\{hx+k(1-x)\}^{2\sigma}}\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right.\right] . S^{w}_{\lambda}\left[\frac{\{hkx(1-x)\}^{\mu}}{\{hx+k(1+x)\}^{2\mu}}\right]$$

$$. F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\frac{z_r\{hkx(1-x)\}^{\sigma_1}}{\{hx+k(1-x)^{2\sigma_1}}, \ldots, \frac{z_r\{hkx(1-x)\}^{\sigma_r}}{\{hx+k(1-x)^{2\sigma_r}}\right) dx$$

$$= \sum_{\alpha=0}^{[\lambda/w]}\sum_{g=1}^{m}\sum_{s=0}^{\infty}(-1)^s\,\phi(\eta_s)\,z^{\eta_s}\{s!\,F_g\}^{-1}\frac{(-\lambda)_{w\alpha}}{\alpha!}L_{\lambda,\alpha}\,\Pi\,\Gamma(\rho+\sigma\eta_s+\mu\alpha)$$

$$.\ \Gamma\{\rho+\sigma\eta_s+\mu\alpha+(1-u-v)/2\}\,\Gamma\{(u+v+1)/2\} . 2^{1-2\rho-2\sigma\eta_s-2\mu\alpha} .$$

$$. \Big[(hk)^{\rho}\,\Gamma\{(1+u)/2\}\,\Gamma\{(1+v)/2\}\,\Gamma\{\rho+\eta_s$$

$$+\mu\alpha+(1-u)/2\}\,\Gamma\{\rho+\sigma\eta_s+\mu\alpha+(1-v)/2]^{-1}$$

$$.\ F^{A+2:B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}], [\rho+\sigma\eta_s+\mu\alpha:\sigma_1,\ldots,\sigma_r],\\ [(c):\psi',\ldots,\psi^{(r)}], [\rho+\sigma\eta_s+\mu\alpha+(1-u)/2:\sigma_1,\ldots,\sigma_r],\end{matrix}\right.\ [\rho+\sigma$$

$$\left.[\rho+\sigma\eta_s+\mu\alpha+(1-v)/2:\sigma_1,\ldots,\sigma_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \quad z_1(4)^{-\sigma_1},\ldots,z_r(4)^{-\sigma_r}\right), \qquad \ldots (3.1)$$

provided that $\sigma > 0$, $Re(\rho+(1-u-v)/2+\sigma f_l/F_l) > 0$, $l = 1, \ldots, m$, $|\arg z| < T\pi/2$, $T > 0$, h, k are non-zero constants and $\{hx+k(1-x)\}$ is non-zero when $0 \le x \le 1$, w is an arbitrary postive constant and the coefficients $L_{\lambda,\alpha}$ $(\lambda, \alpha \ge 0)$ are arbitrary constants real or complex and the resulting multiple series converges.

$$\int_0^1 x^{l-1}(1-x)^{\rho-v}\{hx+k(1-x)\}^{-2\rho+v-1}$$

$$_2F_1\big[u, 1-u; v; hx\{hx+k(1-x)\}^{-1}\big]$$

$$H^{m,n}_{p,q}\left[\frac{z\{hkx(1-x)\}^{\sigma}}{\{hx+k(1-x)\}^{2\sigma}}\left|\begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right|\right] . S^{w}_{\lambda}\left[\frac{\{hkx(1-x)\}^{\mu}}{\{hx-k(1-x)\}^{2\mu}}\right]$$

$$F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\frac{z_1\{hkx(1-x)\}^{\sigma_1}}{\{hx+k(1-x)\}^{2\sigma_1}}, \ldots, \frac{z_r\{hkx(1-x)\}^{\sigma_r}}{\{hx+k(1-x)\}^{2\sigma_r}}\right) dx$$

$$= \sum_{\alpha=0}^{[\lambda/w]}\sum_{g=1}^{m}\sum_{s=0}^{\infty}(-1)^s\,\phi(\eta_s)\,z^{\eta_s}\{s!\,F_g\}^{-1}\frac{(-\lambda)_{w\alpha}}{\alpha!}L_{\lambda,\alpha}(hk)^{\sigma\eta_s+\mu\alpha}$$

$$. \ \pi \ 2^{1-2\rho-2\sigma\eta_s-2\mu\alpha}\,\Gamma(v)\,\Gamma(\rho+\sigma\eta_s+\mu\alpha)\,\Gamma(\rho+\sigma\eta_s+\mu\alpha-v+1)$$

$$. \ \Big[\,\Gamma\{(u+v)/2\}\ \Gamma\{(1-u+v)/2\}\ \Gamma\{\rho+\sigma\eta_s+\mu\alpha+(u-v+1)/2\} \cdot \Gamma\{\rho+\sigma\eta_s+\mu\alpha-(u+v)/2\}\Big]^{-1}$$

$$. \ F^{A+2:B',\ldots,B^{(r)}}_{C+2:D',\ldots,D^{(r)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\ [\rho+\sigma\eta_s+\mu\alpha:\sigma_1,\ldots,\sigma_r], \\ [(c):\psi',\ldots,\psi^{(r)}],\ [\rho+\sigma\eta_s+\mu\alpha+(u-v+1)/2;\sigma_1,\ldots,\sigma_r],\end{matrix}\right.$$

$$[\rho+\sigma\eta_s+\mu\alpha-v+1:\sigma_1,\ldots,\sigma_r]:[(b'):\phi';\ldots;[(b^{(r)}):\phi^{(r)}];$$

$$\left.[\rho+\sigma\eta_s+\mu\alpha-(u+v)/2:\sigma_1,\ldots,\sigma_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \quad z_1\frac{(hk)^{\sigma_1}}{4^{\sigma_1}},\ldots,z_r\frac{(hk)^{\sigma_r}}{4^{\sigma_r}}\right), \qquad \ldots (3.2)$$

provided that $\sigma > 0$, $\mathrm{Re}(v) > 0$, $\mathrm{Re}(\rho + \sigma f_l / F_l) > l = 1, \ldots, m$, $|\arg z| < T\pi/2$, $T > 0$, h, k are non-zero constants and $\{hx + k(1-x)\}$ is non-zero where $0 \le x \le 1$, w is an arbitrary postive constants and the coefficients $L_{\lambda,\alpha}$ $(\lambda, \alpha \ge 0)$ are arbitrary constants real or complex and the resulting multiple series converges. Similarly the result in (2.3) can be reduced to an integral involving the Lauricella's functions as a special case.

(ii) When $\lambda = 0$, the integrals in (2.1) to (2.3), (3.1) and (3.2) reduce to the integrals recently obtained by Chaurasia [2].

The importnace of our integrals possess mainifold generality. At the outset, we recall that in view of the generality of the polynomials $S^{w}_{\lambda}[x]$, on suitable specializing the coefficients $L_{\lambda,\alpha}$, and making a free use of the special cases of $S^{w}_{\lambda}[x]$ listed by Srivastava and Singh [9], our integrals can be reduced to a large number of integrals involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in Fox's H-function and in the multivariable H-function, we can obtain from our results, several integrals involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H functions of one and several variables. Thus our results would at once yield a very large number of integrals involving a large variety of polynomials and various special functions occurring in the problems of mathmetical analysis, applied mathematics and mathematical physics.

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REFERENCES

[1] B.L.J.Braaksma, Asymptotic expansions and analytic continuations for a class of barnes-integrals *Compositio Math.* **15** (1963), 339-341.

[2] V.B.L.Chaurasia, Integration of certain products associated with the multivariable H-function, *Ganita Sandesh* **2** (1988), 66-69.

[3] C.Fox, The *G* and *H* functions as symmetrical Fourier kernels, *Trnas. Amer. Math. Soc.* **98** (1961), 395-429.

[4] P. Skibiński, Some expansion theorems for the H-function, *Ann. Polon. Math.* **23** (1970), 123, 123-138.

[5] H.M.Srivastava, A contour integral involving Fox's H- function, *Indian J. Math.* **14** (1972), 1-6.

[6] H.M.Srivastava and R.Panda, Expansion theorems for the **H** function of several complex variables, *J.Reine Anges. Math.* **288** (1976), 129-138.

[7] H.M.Srivastava and R.Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials. *J.Reine Angew. Math* **283/284** (1976), 265-274.

[8] H.M.Srivastava, K.C. Gupta and S.P.Goyal, *The H-Functions of One and Two Variables with Applications,* South Asian Publishers, New Delhi and Madras, 1982.

[9] H.M.Srivastava and N.P.Singh, The integration of certain products of the multivariable H-function with general class of polynomnials, *Rend.Circ.Mat. Palermo* (2) **32** (1983), 157-187.

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(Dedicated to Professro S.A. Naimpally on his 62th Birthday)

SOME INTEGRALS INVOLVING A GENERAL CLASS OF POLYNOMIALS, FOX'S *H*-FUNCTION AND THE MULTIVARIABLE *H*-FUNCTION

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ABSTRACT

In the present paper, we evaluate four finite integrals involving the products of Fox's *H*-function, the multivariable *H*-function and a general class of polynomials. On account of the most general nature of the functions involved herein a very large number of known and new integrals involving simpler special functions and orthogonal polynomials follows as particular cases of our main integrals.

1. INTRODUCTION

Srivastava [5] introduced the genral class of polynomials (see also Srivastava and Singh [9]) :

$$S_n^m [x] = \sum_{l=0}^{n/m} \frac{(-n)_{ml}}{l\,!} A_{n,l}\, x^l \ , \ l = 0, 1, 2, \ldots \qquad \ldots (1.1)$$

where m is an arbitrary postive integer and the coefficients $A_{n,l}\,(n, l \geq 0)$ are arbitrary constants real or complex. By suitable specializing the coefficients $A_{n,l}$, the general claass of polynomials can be reduced to a large spectrum of polynomials as cited in the paper's referred to above.

The series representation of Fox's H-function (see [4] and [1])

$$H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right] = \sum_{G=0}^{M} \sum_{g=1}^{\infty} (-1)^G \phi(\eta_G)\, z^{\eta_G} \left\{G\,!\,F_g\right\}^{-1} , \qquad \ldots (1.2)$$

where

$$\phi(\eta_G) = \prod_{j=1, j \neq g}^{M} \Gamma(f_j - F_j \eta_G) \prod_{j=1}^{N} \Gamma(1 - e_j + E_j \eta_G)$$

$$\cdot \left\{ \prod_{j=M+1}^{Q} \Gamma(1-f_j+F_j\,\eta_G) \prod_{j=N+1}^{P} \Gamma(e_j-E_j\,\eta_G) \right\}^{-1} \qquad \dots (1.3)$$

and

$$\eta_G = (f_g = G)/F_g\ .$$

For the multivariable H-function defined by Srivastava and Panda ([5] and [7]; see also [6]), we establish four finite integrals in the next section.

2. THE MAIN INTEGRALS

The following integrals have been derived in this section :

$$\int_s^t (x-s)^{u-1}(t-x)^{v-1}(x-w)^{-u-v}\, S_n^m\left[z\left(\frac{x-s}{x-w}\right)^{h}\left(\frac{t-x}{x-w}\right)^{k}\right]$$

$$\cdot\, H_{P,Q}^{M,N}\left[y\left(\frac{x-s}{x-w}\right)^{h'}\left(\frac{t-x}{x-w}\right)^{k'}\left|\begin{matrix}(e_p, E_p)\\(f_Q, F_Q)\end{matrix}\right.\right]$$

$$\cdot\, H_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[(a):e',\dots,e^{(r)}]\ :\\ [(c):\psi',\dots,\psi^{(r)}]\ :\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]\ ;\ z_1\left(\dfrac{x-s}{x-w}\right)^{h_1}\left(\dfrac{t-x}{x-w}\right)^{k_1}\\ \vdots\\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}]\ ;\ z_r\left(\dfrac{x-s}{(x-w)}\right)^{h_r}\left(\dfrac{t-x}{x-w}\right)^{k_r}\end{matrix}\right)dx$$

$$= \sum_{l=1}^{n/m}\frac{(-n)_{ml}}{l\,!}A_{n,l}\,z^l\sum_{g=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^G}{G\,!\,F_g}\,\phi(\eta_G)\,y^{\eta_G}$$

$$\cdot\,(t-s)^{u+v-1+(h+k)l+(h'+k')\eta_G}(t-w)^{-u-hl-h'\eta_G}(s-w)^{-v-kl-k'\eta_g}$$

$$H_{A+2,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda+2\ :(u',v');\dots;(u^{(r)},v^{(r)})}$$

$$\left(\begin{matrix}[1-u-hl-h'\eta_G:h_1,\dots,h_r]\\ [1-u-v-(h+k)\,l-(h'+k')\,\eta_G:h_1+k_1,\dots,h_r+k_r]\end{matrix}\right.$$

$$[1-v-kl-k'\eta_G;k_1,\dots,k_r],[(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]\ ;$$

$$[(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}]\ ;$$

$$\left.\begin{matrix}z_1\left(\dfrac{t-s}{t-w}\right)^{h_1}\left(\dfrac{t-s}{s-w}\right)^{k_1}\\ \vdots\\ z_r\left(\dfrac{t-s}{t-w}\right)^{h_r}\left(\dfrac{t-s}{s-w}\right)^{k_r}\end{matrix}\right) . \qquad \dots (2.1)$$

where $h > 0, k > 0, h' > 0, k' > 0, h_i > 0, k_i > 0$,

$$Re(u + h' f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)}) > 0,$$

$$Re\,(v + k' f_L/F_L + \sum_{i=1}^{r} k_i\, d_j^{(i)}/\delta_j^{(i)}) > 0, w < s < t\,,$$

$$|\, arg\, y \,| < T\pi/2, \ \ |\, arg\,(z_i) | < T_i\, \pi/2\,, T > 0, T_i > 0,$$

$$i = 1, ..., r,\ j = 1, ..., u^{(i)}\,, L = 1, ..., M,$$

m is an arbitrary positive integrer and the coefficients $A_{n,l}\,(n, l \geq 0)$ are arbitrary cosntants real or complex.

$$\left(T = \sum_{1}^{M} E_i - \sum_{N+1}^{P} E_i + \sum_{1}^{M} F_i - \sum_{m+1}^{Q} F_i\right).$$

$$\int_0^t x^{u-1}\,(t-x)^{v-1}\, S_n^m\left[yx^h(t-x)^k\right] H_{P,Q}^{M,N}\left[z\, x^{h'}\,(t-x)^{k'} \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_q)\end{matrix}\right]$$

$$.\ H_{A,\,C:\,[B',\,D'];\,...;\,[B^{(r)},\,D^{(r)}]}^{0,\,\lambda:\ (u',\,v');\,...;\,(u^{(r)},\,v^{(r)})} \left(\begin{matrix}[(a) : e', ..., e^{(r)}] : \\ [(c) : \psi', ..., \psi^{(r)}] :\end{matrix}\right.$$

$$\left.\begin{matrix}[\,(b') : \phi^1]\,; ...;\, [\,(b^{(r)}) : \phi^{(r)}]\,; z_1\, x^{h_1}\,(t-x)^{k_1} \\ \vdots \\ [\,(d') : \delta^1\,]\,; ...;\, [\,(d^{(r)} : \delta^{(r)}]\,; z_r\, x^{h_r}\,(t-x)^{k_r}\end{matrix}\right) dx$$

$$= \sum_{l=0}^{n/m} \frac{(-n)_{ml}}{l\,!}\, A_{n,l}\, y^l \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-1)^G}{G!\, F_g}\, \phi(\eta_G) z^{\eta_G}\, t^{u+v+(h+k)\,l\,(h'+k')\,\eta_G - 1}$$

$$.\ H_{A+2,\,C+1\,:\ [B',\,D'];\,...;\,[B^{(r)},\,D^{(r)}]}^{0,\quad \lambda+2\,:\,(u',\,v');\,...;\,(u^{(r)},\,v^{(r)})} \left(\begin{matrix}[1 - u - hl - h'\,\eta_G : h_1\,, ..., h_r], \\ [l - u - v - (h+k)l - (h'+k')\eta_G :\end{matrix}\right.$$

$$[1 - v - kl - k'\eta_G : k_1\,; ...; k_r]\,, [(a) : e'; ...\, e^{(r)}] : [(b') : \phi'\,]\,; ...; [b^{(r)}) : \phi^{(r)}] :$$

$$h_1 + k_1\,; ...; h_r + k_r], \qquad [(c) \!: \psi'\,; ...; \psi^{(r)}] : [(d') : \delta'\,]\,; ...; [(a^{(r)}) : \delta^{(r)}] :$$

$$\left.\begin{matrix}z_1\, t^{h_1 + k_1} \\ \vdots \\ z_r\, t^{h_r + k_r}\end{matrix}\right). \qquad ... (2.2)$$

where

$h > 0, k > 0, h' > 0, k' > 0, h_i > 0, k_i > 0,\ |\, arg\, z\,| < T\,\pi/2,$

$$|\,arg\,(z_i)\,| < T_i\,\pi/2\,,\; Re\left(u + h'\,f_L/F_L + \sum_{i=1}^{r} h_i\; d_j^{(i)}/\delta_j^{(i)}\right) > 0,$$

$$Re\left(v + k'\,f_L/F_L + \sum_{i=1}^{r} k_i\; d_j^{(i)}/\delta_j^{(i)}\right) > 0,\; i = 1, \ldots, r,\; j = 1, \ldots, u^{(i)}\,,$$

$L = 1,\ldots,M$, m is an arbitrary postive integers and the coefficients $A_{n,l}\,(n, l \geq 0)$ arbitrary constants are real or complex.

$$\int_0^1 x^{u-1}(1-x)^{v-1}\,{}_2F_1(\alpha, \beta : u : x) S_n^m[y(1-x)^h]\, H_{P,Q}^{M,N}\left[z(1-x)\,h'\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right]$$

$$.\; H_{A,\,C:\,(B',\,D'];\,\ldots;\,[B^{(r)},\,D^{(r)}];}^{0,\,\lambda:\;(u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}\;\begin{matrix}[(a) : e',\ldots, e^{(r)}] \;:\\ [(c) : \psi',\ldots, \psi^{(r)} \;:\end{matrix}$$

$$\left.\begin{matrix}[(b') : \phi'] \;;\ldots;\; [(b^{(r)}) : \phi^{(r)}] \;;\; z_1(l-x)^{h_1}\\ \vdots \\ [(d') : \delta'] \;;\ldots;\; [(d^{(r)}) : \delta^{(r)}] \;;\; z_r(1-x)^{h_r}\end{matrix}\right)dx$$

$$= \sum_{l=0}^{n/m} \frac{(-n)_{ml}}{l\,!}\, A_{n,l}\, Y^l \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-1)^G}{G\,!\,F_g}\; \phi\,(\eta_G)\, z^{\eta_G}$$

$$.\;\Gamma\,(u)\, H_{A+2,\,C+1\,:\,[B',\,D'];\,\ldots;\,[B^{(r)},\,D^{(r)},\,D^{(r)}]}^{0,\,\lambda+1\;:\,(u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}$$

$$\left(\begin{matrix}[1 - v - hl - h'\eta_G : h_1\,,\ldots, h_r]\\ [1 + \alpha - u - v - hl - h'\,\eta_G : h_1, \ldots, h_r]\end{matrix}\right.$$

$$[1 - u - v - hl - h'\,\eta_G + \alpha + \beta : h_1\,;\ldots;\, h_r],\; [(a) : e',\ldots, e^{(r)}] :$$

$$[1 + \beta - u - v - hl \;- h'\eta_G : h_1\,,\ldots, h_r],\; [(c) : \psi',\ldots \psi^{(r)}] \;:$$

$$\left.\begin{matrix}[(b') : \phi'] \;;\ldots;\; [(b^{(r)} : \phi^{(r)}] \;:\; z_1\\ \vdots \\ [(d') : \delta'] \;;\ldots;\; [(d^{(r)} : \delta^{(r)}] \;:\; z_r\end{matrix}\right), \qquad \ldots (2.3)$$

where

$h > 0, h' > 0, h_i > 0, T > 0, T_i > 0$, $|\arg z| < T\,\pi/2$, $|\arg\,(z_i)| < T_i\,\pi/2$,

$$Re\,(u + v - \alpha - \beta) > 0,\; Re\,(u) > 0,\; Re\left(v + h'\,f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{i}/\delta_j^{(i)}\right) > 0,$$

$i = 1,\ldots,r$, $j = 1,\ldots, u^{(i)}$, $l = 1,\ldots,M$, m is an arbitrary poistive integer and the coefficients $A_{n,l}\,(n, l \geq 0)$ are arbitrary constants real or complex.

$$\int_{-1}^{1} (1+x)^{u-1}\,(1-x)^{v-1}\, P_w^{(\alpha,\,\beta)}\,(1 - \frac{st}{2}\,(1-x))\; S_n^m\left[y\,(1-x)^h\,(1+x)^k\right]$$

$$\cdot\, H_{P,Q}^{M,N}\left[z\,(1+x)^{h'}\,(1-x)^{k'} \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]$$

$$\cdot\, H_{A,\,C\,:\,[B',\,D'];\,\ldots;\,[B^{(r)},\,D^{(r)}]}^{0,\,\lambda\,:\,(u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}\left(\begin{matrix}[(a):e',\ldots,e^{(r)}]\;:\\ [(c):\psi',\ldots,\psi^{(r)}]\;:\end{matrix}\right.$$

$$\left.\begin{matrix}[b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]\,;\, z_1\,(1+x)^{h_1}\,(1-x)^{k_1}\\ \vdots\\ [(d'):\delta'];\ldots;[(d)^{(r)}):\delta^{(r)}]\,;\, z_r\,(1+x)^{h_r}\,(1-x)^{k_r}\end{matrix}\right) dx$$

$$= \sum_{l=0}^{n/m} \frac{(-n)_{ml}}{l\,!} A_{n,l}\; y^l \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-1)^G}{G\,!\,F_g}\, \phi(\eta_G)\; z^{\eta_G}$$

$$\cdot\, \frac{2^{u+v+(h+k)l+(h'+k')^{\eta}G-1},\ (\alpha+1:w)}{w!} \sum_{R=0}^{w} \frac{(-w:R)\,(1+\alpha+\beta+w:R)}{R!\,(\alpha+1:R)} \left(\frac{st}{2}\right)^R$$

$$\cdot\, H_{A+2,\,C+1\,:\,[B',\,D'];\,\ldots;\,[B^{(r)},\,D^{(r)}]}^{0,\;\lambda+2\;:\,(u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}\left(\begin{matrix}[1-u-hl-h'\eta_G : h_1,\ldots,h_r],\\ [1-u-v-(h+k)l-(h'+k')\end{matrix}\ \eta_G - R\right.$$

$$\begin{matrix}[1-v-kl-k'\eta_G-R : k_1;\ldots;k_r],\ [(a):\theta',\ldots,\theta^{(r)}]\;:\\ :h_1+k_1\;;\ldots;h_r+k_r], \qquad\qquad [(c):\psi',\ldots,\psi^{(r)}]\;:\end{matrix}$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]\;;\; z_1^2 h_1+k_1\\ \vdots\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\;;\; 2^{h_r+k_r}\end{matrix}\right). \qquad \ldots (2.4)$$

where

$$h>0,\ k>0,\ h'>0,\ k'>0,\ h_i>0,\ k_i>0,\ T>0,\ T_i>0,$$

$$|\arg z| < T\pi/2,\ |\arg(z_i)| < T_i\,\pi/2,\ Re(u+h'f_l/F_L+\sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)})>0,$$

$$Re\ (v+k'\,f_L/F_L+\sum_{i=1}^{r} k_i\; d_j^{(i)}/\delta_j^{(i)}>0,\ i=1,\ldots,r,\ j=1,\ldots,u^{(i)},$$

$L = 1,\ldots,M$, m is an arbitrary positive integer and the coefficients $A_{n,l}\,(n, l \geq 0)$ an arbitrary constants real or complex and the series on the right converages.

Proof : To establish (2.1), express the general class of polynomials occuring in integrand (2.1) in series form given by (1.1), and express the Fox's H-function in series with the help of (1.2), and then interchanged the order of summation and integration (which is permissible under the conditions stated above), evaluate the inner integral with the help of a result of Chaurasia ([3],p.210,eq.(2.1)), we arrive at the desired result.

The proofs of the integrals (2.2) to (2.4) can be developed by proceeding on similar lines with the help of known integrals ([2],p.85,eq.(2.6),eq.(2.4) and p.84,eq.(2.2)).

3 SPECIAL CASES

(i) When $n = 0$, the integrals in (2.1) to (2.4) reduce to the following integrals

$$\int_s^t (x-s)^{u-1}\,(t-x)^{v-1}\,(x-w)^{-u-v}\; H_{P,Q}^{M,N}\left[y\left(\frac{x-s}{x-w}\right)^{h'} \left(\frac{t-x}{x-w}\right)^{k'} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\cdot\, H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}]: \end{matrix} \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]\;;\; z_1\left(\frac{x-s}{x-w}\right)^{h_1}\left(\frac{t-x}{x-w}\right)^{k_1} \\ \vdots \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\;;\; z_r\left(\frac{x-s}{x-w}\right)^{h_r}\left(\frac{t-x}{x-w}\right)^{k_r} \end{matrix} \right) dx$$

$$= \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-1)^G}{G!\,F_g}\,\phi(\eta_G)\, Y^{\eta_G}\,(t-S)^{u+v-1+(h'+k')\eta_G}\,(t-w)^{-u-h'\eta_G}$$

$$\cdot\,(s-w)^{-v-k'\eta_G}\, H_{A+2,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\;\lambda+2\;:(u',v');\ldots;(u^{(r)},v^{(r)})}$$

$$\left(\begin{matrix} [1-u-h'\eta_G:h_1,\ldots,h_r],\,[1-v-k'\eta_G:k_1,\ldots,k_r],\,[(a):\theta',\ldots,\theta^{(r)}]: \\ \\ [1-u-v-(h'+k')\eta_G:h_1+k_1;\ldots;h_r+k_r],\qquad [(c):\psi',\ldots,\psi^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]\,;\,z_1\left(\frac{t-s}{t-w}\right)^{h_1}\left(\frac{t-s}{s-w}\right)^{k_1} \\ \vdots \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)})\,;\,z_r\left(\frac{t-s}{t-w}\right)^{h_r}\left(\frac{t-s}{s-w}\right)^{k_r} \end{matrix} \right), \qquad \ldots (3.1)$$

valid under the conditions obtainable from (2.1)

$$\int_0^t x^{u-1}(t-s)^{v-1}\, H^{M,N}_{P,Q}\left[z\,x^{h'}(t-x)^{k'}\left|\begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right.\right]$$

$$.\; H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]\;:\\ [(c)\;:\;\psi',\ldots,\psi^{(r)}]\;:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)},\phi^{(r)}]\;;\; z_1 x^{h_1}(t-x)^{k_1}\\ \vdots\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\;;\; z_r x^{h_r}(t-x)^{k_r}\end{matrix}\right)dx$$

$$= \sum_{g=1}^{M}\sum_{G=0}^{\infty}\frac{(-1)^G}{G!\,F_g}\,\phi(\eta_G)\, z^{\eta_G}\, t^{u+v+(h'+k')\eta_G-1}$$

$$H^{0,\;\lambda+2\;:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+2,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[1-u-h'\eta_G:h_1,\ldots,h_r]\;:\\ [1-u-v-(h'+k')\eta_G:\end{matrix}\right.$$

$$[1-v-k'\eta_G:k_1,\ldots,k_r],\;[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]\;;$$

$$h_1+k_1,\quad\ldots,h_r+k_r]\quad,\;[(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots,[(d^{(r)}):\delta^{(r)}]\;;$$

$$\left.\begin{matrix}z_1 t^{h_1+k_1}\\ \vdots\\ z_r t^{h_r+k_r}\end{matrix}\right), \qquad \ldots (3.2)$$

valid under the conditions obtainable from (2.2)

$$\int_0^1 x^{u-1}(1-x)^{v-1}\,{}_2F_1(\alpha,\beta;u:x)\,H^{M,N}_{P,Q}\left[z(1-x)^{h'}\left|\begin{matrix}(e_p, E_p)\\ (f_Q, E_Q)\end{matrix}\right.\right]$$

$$H^{0,\lambda:\;(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;\\ [(c):\psi',\ldots,\psi^{r}]:[(d'):\delta'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(r)}):\phi^{(r)}]\;;z_1(1-x)^{h_1}\\ [(d^{(r)}):\delta^{(r)}]\;;z_r(1-x)^{h_r}\end{matrix}\right)dx$$

$$= \sum_{g=1}^{M}\sum_{G=0}^{\infty}\frac{-1)^G}{G!\,F_g}\,\phi(\eta_G)\,z^{\eta_G}\,\Gamma(u)\,H^{0,\lambda+1\;:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+2,C+1:(B',D');\ldots;[B^{(r)},D^{(r)}]}$$

$$[1-v-h'\eta_G:h_1,\ldots,h_r],\;[1-u-v-h'\eta_G+\alpha+\beta:h_1,\ldots,\;[(a):e',\ldots,e^{(r)}]:$$

$$[1+\alpha-u-v-h'\eta_G:h_1,\ldots,h_r],\;[1+\beta-u-v-h'\eta_G:h_1,\ldots,h_r],\;[(c):\psi',\ldots,\psi^{(r)}]:$$

$$\begin{matrix}[(b') : \psi^{2}] ; \ldots ; [(b)^{(r)} : \phi^{(r)}] ; z_1 \\ [(d') : \delta'] ; \ldots ; [(a^{(r)}) : \delta^{(r)}] ; z_r\end{matrix} \qquad \ldots (3.3)$$

valid under the conditions obtainable from (2.3).

$$\int_{-1}^{1} (1+x)^{u-1} (1-x)^{v-1} P_{w}^{(\alpha,\beta)} \left(1 - \frac{st}{2}(1-x)\right) H_{P,Q}^{M,N}\left[z\,(1+x)^{h'} (1-x)^{k'} \left| \begin{matrix} (e_p , E_p) \\ (f_Q , F_Q) \end{matrix} \right. \right]$$

$$H_{A,C : [B',D'] ; \ldots ; [B^{(r)},D^{(r)}]}^{0,\lambda : (u',v') ; \ldots ; (u^{(r)},v^{(r)})} \left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] ; [(b') : \phi'] ; \ldots ; \\ [(c) : \psi', \ldots, \psi^{(r)}] ; [(d') : \delta'] ; \ldots ; \end{matrix} \begin{matrix} [(b^{(r)}) : \phi^{(r)}] ; z_1 (1+x)^{h_1} (1-x)^{k_1} \\ \vdots \\ [(d^{(r)}) : \delta^{(r)}] ; z_r (1+x)^{h_r} (1-x)^{k_r} \end{matrix} \right) dx$$

$$= \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-1)^G}{G!\,F_g} \phi(\eta_G) \frac{2^{u+v+(h'+k')\eta_G - 1}, (\alpha+1 : w)}{w!}$$

$$\cdot \sum_{R=0}^{w} \frac{(-w : R)\,(1+\alpha+\beta+w : R)}{R!\ (\alpha+1 : R)} \left(\frac{st}{2}\right)^{R}$$

$$H_{A+2,C+1 : [B',D'] ; \ldots ; [B^{(r)},D^{(r)}]}^{0,\lambda+2 : (u',v') ; \ldots ; (u^{(r)},v^{(r)})} \left(\begin{matrix} [1-u-h'\eta_G : h_1, \ldots, h_r] , \\ [1-u-v-(h'+k')\eta_G - R : \end{matrix} \right.$$

$$\begin{matrix} [1-v-k'\eta_G - R : k_1, \ldots, k_r] , [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ; \ldots ; \\ h_1+k_1 ; \ldots ; h_r+k_r] \qquad , [(c) : \psi', \ldots, \psi^{(r)}] : [(d') : \delta'] ; \ldots ; \end{matrix}$$

$$\left. \begin{matrix} [(b^{(r)}) : \phi^{(r)}] ; z_1\,2^{h_1+k_1} \\ \vdots \\ [(d^{(r)}) : \delta^{(r)}] ; z_r\,2^{h_r+k_r} \end{matrix} \right) , \qquad \ldots (3.4)$$

valid under the conditions obtainable form (2.4).

(*ii*) Taking n=0 and leting $h' \to 0, k' \to 0$, the integrals in (2.1), (2.2) and (2.4) reduce to the integrals obtained by Chaurasia ([3],p.210,eq.(2.1); [2],p.85,eq.(2.6), p.84,eq.(2.1)).

(*iii*) Leting $h' \to 0$ and n = 0, the integral in (2.3) reduces to an integral obtained by Chaurasia ([2],p.85,eq.(2.4)).

The importance of our results lies in its manifold generality. In view of the generality of the polynomials $S_n^m [x]$, on suitably specializing the coefficients $A_{n,l}$, and making a free use of the special cases of $S_n^m [x]$ listed by Srivastava and Singh [9], our resluts can be reduced to a large number of integrals (and also its particular cases) involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in the multivariables H-funcion, we can obtain from our integrals, several integrals, involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E,F,G and H-functions of one and several variables. Thus the integrals presented in this paper would at once yield a very large number of results involving a large variety of polynomials and various special functions occurring in the literature.

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REFERENCES

[1] B.L.J.BRAAKSMA, Asymptotic expansions and analytic continuations for a class of Barnes-integral, *Compositio Math.* **15** (1963), 239-341.

[2] V.B.L.CHAURASIA, Some integrals associated with the H- function of several complex variables, *Kyungpook Math.* J. **20** (1980), 83-86.

[3] V.B.L.CHAURASIA, An integral involving H-function of two variables, Indian J.Math. **21** (1979), 209-212.

[4] C.FOX, The G and H functions as symmetrical Fourier kernels, *Trnas. Amer. Math. Soc.* **98** (1961), 395-429.

[5] H.M.SRIVASTAVA, A Contour integral involving Fox's H- function, *Indian J.Math.* **14** (1972), 1-6.

[6] H.M.SRIVASTAVA, K.C.GUPTA AND S.P.GOYAL, *The H-Functions of One and Two Variables with Applications,* South Asian Publishers, New Delhi and Madras, 1982.

[7] H.M.SRIVASTAVA AND R.PANDA, Expansion theorems for the H function of several complex variables, *J.Reine Angew, Math.* **288** (1976), 129-145.

[8] H.M.SRIVASTAVA AND R.PANDIA, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J.Reine Angew. Math.* **283/284** (1976), 265-274.

[9] H.M.SRIVASTAVA AND N.P.SINGH, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ.Mat.Palermo* **(2) 32** (1983), 157-187.

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REFERENCES

[1] B.L.J. BRAAKSMA, Asymptotic expansions and analytic continuations for a class of Barnes-integral, *Compositio Math.* 15 (1963), 239-341.

[2] [illegible], Some integrals associated with the H-function of several complex variables, *Kyungpook Math. J.* 20 (1980), [illegible]

[3] [illegible], An integral involving H-function of two variables, *Indian J. Math.* 21 (1979), [illegible]

[4] C. FOX, The G and H functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.* 98 (1961), 395-429.

[5] H.M. SRIVASTAVA, A Contour integral involving Fox's H-function, *Indian J. Math.* 14 (1972), 1-6.

[6] H.M. SRIVASTAVA, K.C. GUPTA AND S.P. GOYAL, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[7] H.M. SRIVASTAVA AND R. PANDA, Expansion theorems for the H function of several complex variables, *J. Reine Angew. Math.* 288 (1976), 129-145.

[8] H.M. SRIVASTAVA AND R. PANDA, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* 283/284 (1976), 265-274.

[9] H.M. SRIVASTAVA AND N.P. SINGH, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* (2) 32 (1983), 157-187.

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(Dedicated to Professor S.A.Naimpally on his 62th Birthday)

THE INTEGRATION OF CENRTAIN PRODUCTS OF THE MULTIVARIABLE *H*-FUNCTION WITH FOX'S *H*-FUNCTION AND A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

This paper presents six finite integrals involving the product of Jacobi Polynomials, Fox's *H*-function, the multivariable *H*- function and a general class of polynomials. The results established here are quite general in nature and a large number of (known and new) integrals can be obtained by specializing the parameters suitably of the various function involved in them.

1. INTRODUCTION

Srivastava and Panda [7] defined the multivariable H-function as (*see also* [8])

$$H\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\begin{pmatrix} [(a):e';\ldots;e^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)});z_1 \\ [(c):\psi';\ldots;\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];z_r \end{pmatrix}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} V(\xi_1,\ldots,\xi_r)\,U_1(\xi_1)\ldots U_r(\xi_r)\,z_1^{\xi_1}\ldots z_r^{\xi_r}\,d\xi_1\ldots d\xi_r \qquad \ldots(1)$$

where $$v(\xi_1,\ldots,\xi_r)=\frac{\prod\limits_{J=1}^{\lambda}\Gamma\left[1-a_j+\sum\limits_{i=1}^{r}\theta_j^{(i)}\,\xi_1\right]}{\prod\limits_{j=\lambda+1}^{A}\Gamma\left[a_j-\sum\limits_{i=1}^{r}\theta_j^{(i)}\,\xi_i\right]\prod\limits_{j=1}^{C}\Gamma\left[1-c_J+\sum\limits_{i=1}^{r}\psi_j^{(i)}\,\xi_i\right]}$$

and

$$U_i(\xi_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma\left[a_j^{(i)} - \delta_j^{(i)} \xi_i\right] \prod_{j=1}^{v^{(i)}} \Gamma\left[1 - b_j^{(i)} + \phi_j^{(i)} \xi_i\right]}{\prod_{j=u^{(i)}+1}^{D^{(i)}} \Gamma\left[1 - d_j^{(i)} + \delta_j^{(i)} \xi_i\right] \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma\left[b_j^{(i)} - \phi_j^{(i)} \xi_i\right]}$$

$$i = 1, \ldots, r \qquad \ldots (3)$$

The multiple integral in (1) converges absolutely if

$$|\, arg\,(Z_i)\,| \; < \frac{1}{2} T_i \pi \;\; i = 1, \ldots, r \qquad \ldots (4)$$

where

$$T_i = -\sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)}$$

$$+ \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, i = 1, \ldots, r \qquad \ldots (5)$$

We shall require the following definition of the general class of polynomials ([9], *p*.1, *Eq*.(1))

$$S_n^m\,[x] = \sum_{\alpha'}^{[n/m]} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'}\, x^{\alpha'}, \alpha' = 0, 1, 2, \qquad \ldots (6)$$

where m is arbitrary positive integer and the coefficients $A_{n,\alpha'}\,(n, \alpha' \geq 0)$ are arbitrary constants, real or complex.

By suitably specializing the coefficients $A_{n,\alpha'}$, the general class of polynomials $S_n^m\,[x]$ can be reduced to the well-known classical orthogonal polynomials such as Hermite polynomials, Jacobi polynomials and its various pecial cases, Legendre polynomals, Tchebycheff polynomials, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials etc. and their various combinations.

The seies representaion of Fox's H-function (see also [5])

$$H_{P,Q}^{M,N}\left[z \left| \begin{matrix} (e_p), E_p) \\ (f_Q, F_Q) \end{matrix} \right. \right] = \sum_{g=1}^{M} \sum_{G=0}^{\infty} (-1)^G \, \phi(\eta_G)\, z^{\eta_G} \left\{G!\, F_g\right\}^{-1}, \qquad \ldots (7)$$

where

$$\phi(\eta_G) = \prod_{j=1, j\neq g}^{M} \Gamma(f_j - F_j \eta_G) \prod_{j=1}^{N} \Gamma(1 - e_j + E_j \eta_G)$$

$$\cdot \left\{ \prod_{j=M+1}^{Q} \Gamma(1 - f_j + F_j \eta_G) \prod_{j=N+1}^{P} \Gamma(e_j - E_j \eta_G) \right\}^{-1} \qquad \ldots (8)$$

and $\eta_G = (f_g + G)/F_g$

The following notations and known results will be used throughout this paper :

$$(\alpha)_w = \frac{\Gamma(\alpha+w)}{\Gamma(\alpha)} = \begin{cases} 1, \text{ is } w = 0 \\ \alpha(\alpha+1)\ldots(\alpha+w-1), \ \forall \ \ w \, \varepsilon \, \{1, 2, 3, \ldots\} \end{cases} \quad \ldots (9)$$

Next, for the Jacobi polynomials $P_w^{(\alpha,\beta)}(x)$ [6, p. 25 4, E_q. (1)], we have

$$P_w^{(\alpha,\beta)}(t+\rho)\, P_w^{(\alpha,\beta)}(t-\rho)\, \frac{(-1)^w (1+\alpha)_w (1+\beta)_w}{(w\,!)^2}$$

$$\cdot \sum_{R=0}^{w} (-w)_R \frac{(1+\alpha+\beta+w)_R}{(1+\alpha)_R (1+\beta)_R} P_R^{(\alpha,\alpha)}(x)\, t^R \quad \ldots (10)$$

$$\rho^w P_w^{(\alpha,\alpha)} \left(\frac{1-xt}{\rho}\right) = \frac{(1+\alpha)_W}{w\,!} \sum_{R=0}^{w} \frac{(-w)_R}{(1+\alpha)_R} P_R^{(\alpha,\alpha)}(x)\, t^R \quad \ldots (11)$$

$$\frac{1}{\rho}(1-t-\rho)^{-\alpha}(1+t+\rho)^{\beta} = 2^{-\alpha-\beta} \sum_{R=0}^{\infty} P_R^{(\alpha,\beta)}(x)\; t^R \quad \ldots (12)$$

In each of the formulas (10), (11) and (12), and throughout this paper, $\rho = (1 - 2xt + t^2)^{1/2}$. Formulas (10), (11) and (12) can be found, for example in [1,p.945] and [3,p.172], respectively.

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{v} P_w^{(\alpha,\beta)}(x)\, S_n^m[(1+x)^h]\, H_{P,Q}^{M,N}\left[z(1+x)^{h'} \middle| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix}\right]$$

$$\cdot\, H \begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\alpha+v+h'\eta_G+h\alpha'+1}\, \Gamma(\alpha+v+1) \sum_{\alpha'=0}^{[n/m]} \sum_{g=1}^{M} \sum_{G=0}^{\infty} \frac{(-n)_{m\alpha'}}{\alpha'\,!} A_{n,\alpha'}$$

$$\cdot \frac{(-1)^G \phi(\eta_G)}{G\,!\,G_g} z^{\eta_G}$$

$$\cdot\, H^{0,\lambda+2\ :\ (u',v');\ \ldots;\ (u^{(r)},v^{(r)})}_{A+2,C+2\ :\ [B',D'];\ \ldots;\ [B^{(r)},D^{(r)}]} \left(\begin{matrix} [-v-h'\eta_G-h\alpha' : h_1;\ \ldots;\ h_r]' \\ [(c) : \psi';\ \ldots;\ \psi^{(r)}], \end{matrix} \right.$$

$$\begin{matrix} [\beta-v-h'\eta_G-h\alpha' : h_1;\ \ldots;\ h_r], [(a) : \theta';\ \ldots;\ \theta^{(r)}] : \\ [\beta-v-h'\eta_G+w : h_1;\ \ldots;\ h_r], [-\alpha-v-h'\eta_G-h\alpha'-w-1 : h_1;\ \ldots;\ h_r] : \end{matrix}$$

$$\left. \begin{matrix} [(b') : \phi'];\ \ldots;\ [(b^{(r)}) : \phi^{(r)}];\ 2^{h_1}z_1 \\ [(d') : \delta'];\ \ldots;\ [(d^{(r)}) : \delta^{(r)}] : 2^{h_r} z_r \end{matrix} \right) \quad \ldots (13)$$

where

$Re(\alpha) > -1$, $Re(v) > -1$, $h > 0$, $h' > 0$, $h_i > 0$, $|\, arg\, z \,| < \dfrac{T\pi}{2}$, $|\,\arg\,|\,(z_i)\,| < \dfrac{T_i\pi}{2}$,

$T > 0$, $T_i > 0$, $\mathrm{Re}\left(v + h' f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)}\right) > -1$, $i = 1, \ldots, r$,

$j = 1, \dots, u^{(i)}$, $L = 1, \dots, M$, and m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}\,(n, \alpha' \geq 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} (1-x)^{u}(1+x)^{v} P_{w}^{(\alpha,\beta)}(x)\; S_{n}^{m}\left[(1-x)^{h}(1+x)^{k}\right]$$

$$H_{P,Q}^{M,N}\left[z\,(1-x)^{h'}\,(1+x)^{k'}\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right].\,H\begin{pmatrix}z_1\,(1-x)^{h_1}(1+x)^{k_1}\\ \vdots \\ z_r\,(1-x)^{h_r}(1+)^{k_r}\end{pmatrix}dx$$

$$= 2^{u+v+(h'+k')\eta_G+(h+k)\alpha'+1}\sum_{\alpha'=0}^{[n/m]}\sum_{g=1}^{M}\sum_{G=0}^{\infty}\sum_{l=0}^{w}\frac{(-n)_{m\alpha'}}{\alpha'!}A_{n,\alpha'}$$

$$.\,\frac{(-1)^{G}\phi(\eta_G)}{G!\,F_g}\,z^{\eta_G}\,\frac{(-w)_l\,(\alpha+\beta+w+1)_l}{(\alpha+1)_l\,l!}$$

$$.\,H_{A+2,C+1:[B',D'];[B^{(r)},D^{(r)}]}^{[0,\lambda+2:(u',v');\dots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[-v-h'\eta_G-k\alpha':k_1;\dots;k_r],\\ [(c):\psi';\dots;\psi^{(r)}],\end{matrix}\right.$$

$$[-u-h\alpha'-l:h_1;\dots;h_r],\;[(a):\theta';\dots;\theta^{(r)}]:$$

$$[-u-v-(h'+k')\eta_G-(h+k)\alpha'-l-1:\;h_1+k_1;\dots;h_r+k_r]:$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\;2^{h_1+k_r}z_1\\ \vdots\\ [(d'):\delta'];\dots;[(d)^{(r)}):\delta^{(r)}];\;2^{h_r+k_r}z_r\end{matrix}\right), \qquad \dots (14)$$

where

$Re(u) > -1$, $Re(v) > -1$, $h > 0$, $k > 0$, $h' > 0$, $k' > 0$, $h_i > 0$,

$k_i > 0$, $|\,arg\,(z)\,| < \frac{T\pi}{2}$, $|\,arg\,(z_i)\,| < \frac{T_i\,\pi}{2}$, $T > 0$, $T_i > 0$,

$Re(u + h' f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)} > -1$,

$Re(v + k' f_L/F_L + \sum_{i=1}^{r} k_i\, d_j^{(i)}/\delta_j^{(i)}) > -1$, $i = 1, \dots, r$,

$j = 1, \dots, u^{(i)}$, $L = 1, \dots, M$, and

m is an arbitrary positive integer and the coefficients $A_{n\,.\,\alpha'}$, $(n, \alpha' >= 0)$ are arbitrary constants, real or complex. The integral (14) can be established by making use of a known result recently obtained by Chaurasia and Girdhari Lal in [2,*p*.100,*Eqn*.(1.12)].

2. MAIN INTEGRALS

The following integrals have been evaluated in this section

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{v} P_w^{(\alpha,\beta)}(t+\rho)\, P_w^{(\alpha,\beta)}(t-\rho)\, S_n^m[(1+x)^h]$$

$$\cdot H_{P,Q}^{M,N}\left[z(1+x)^{h'}\left|\begin{matrix}(e_p, E_p)\\(f_Q, F_Q)\end{matrix}\right.\right] H\begin{pmatrix}z_1(1+x)^{h_1}\\ \vdots \\ z_r(1+x)^{h_r}\end{pmatrix} dx$$

$$= 2^{\alpha+v+h'\eta_G+h\alpha'+1}(-1)^w \frac{\Gamma(1+\alpha+w)\,\Gamma(1+\beta+w)}{(w\,!)^2}$$

$$\cdot \sum_{\alpha'=0}^{[n/m]} \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{R=0}^{w} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} \frac{(-1)^G \phi(\eta_G)\, z^{\eta_G}}{G\,!\,F_g}$$

$$\cdot \frac{(-w)_R(1+\alpha+\beta+w)_R}{\Gamma(1+\beta+R)}$$

$$\cdot H_{A+2,\,C+2\,:\,[B',D'];\,\ldots;\,[B^{(r)},D^{(r)}]}^{0,\,\lambda+2\,:\,(u',v');\,\ldots;\,(u^{(r)},v^{(r)})}\left(\begin{matrix}[-v-h'\eta_G-h\alpha' : h_1;\ldots;h_r],\\ [(c):\psi';\ldots;\psi^{(r)}],\end{matrix}\right.$$

$$[\alpha-v-h'\eta_G-h\alpha' : h_1;\ldots;h_r],\ [(a):e';\ldots;e^{(r)}] :$$
$$[\alpha-v-h'\eta_G-h\alpha'+R : h_1;\ldots;h_r],\ [-\alpha-v-h'\eta_G-h\alpha'-R-1 : h_1;\ldots;h_r] :$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b)^{(r)}):\phi^{(r)}];\ 2^{h_1}z_1\\ \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\ 2^{h_r}z_r\end{matrix}\right), \qquad \ldots (15)$$

where

$\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(v) > -1$, $h > 0$, $h' > 0$, $h_i > 0$,

$\mathrm{Re}(v + h'f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)}) > -1$, $|\arg z| < \frac{T\pi}{2}$,

$|\arg(z_i)| < \frac{T_i\pi}{2}$, $T > 0$, $T_1 > 0$, $i = 1, \ldots, r$, $j = 1, \ldots, u^{(i)}$,

$L = 1, \ldots, M$

and m ia an arbitrary positive integer and the coefficients $A_{n,\alpha'}$ $(n, \alpha' \geq 0)$ are arbitrary constant, real or complex.

$$\int_{-1}^{1} \rho^w (1-x)^{\alpha}(1+x)^{v} P_w^{(\alpha,\alpha)}\left(\frac{1-xt}{\rho}\right) S_n^m\left[(1+x)^h\right]$$

$$. H^{M,N}_{P,Q}\left[z\,(1+x)\,h' \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix} \right] H \begin{pmatrix} z_1\,(1+x)^{h_1} \\ z_r\,(1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\alpha+v+h'\eta_G+h\alpha'+1}\, \frac{\Gamma(1+\alpha+w)}{w\,!} \sum_{\alpha'=0}^{[n/m]} \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{R=0}^{w} \frac{(-n)_m\alpha'}{\alpha'\,!}\, A_{n,\alpha'}$$

$$. \frac{(-1)^G\,\phi(\eta_G)}{G\,!\,F_g}\; z^{\eta_G}\,(-w)_R\, t^R$$

$$. H^{0,\ \lambda+2\ :\ (u', v');\,\ldots;\,(u^{(r)}, v^{(r)})}_{A+2,\ C+2\ :\ [B', D'];\,\ldots;\,[B^{(r)}, D^{(r)}]} \left(\begin{matrix} [-v\,h'\eta_G - h\alpha' : h_1\,;\ldots;\,h_r], \\ [(c) : \psi'\,;\ldots;\,\psi^{(\rho)}]\,, \end{matrix} \right.$$

$$\begin{matrix} [\alpha - v - h'\eta_G - h\alpha' : h_1\,;\ldots;\,h_r]\,,\ [(a) : e'\,;\ldots;\,e^{(r)}] : \\ [\alpha - v - h'\eta_G - h\alpha' + R : h_1;\ldots;\,h_r],\ [-\alpha - v - h'\eta_G - h\alpha' - R - 1 : h_1;\ldots;\,h_r] : \end{matrix}$$

$$\left. \begin{matrix} [(b') : \phi']\,;\ldots;\,[(b^{(r)}) : \phi^{(r)})]\,;\, 2^{h_1} z_1 \\ [(d') : \delta']\,;\ldots;\,[(d^{(r)}) : \delta^{(r)}]\,;\, 2^{h_r} z_r \end{matrix} \right), \qquad \ldots (16)$$

where

$\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(v) > -1$, $h > 0$, $h' > 0$, $h_i > 0$,

$\mathrm{Re}(v + h' f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{(i)} / \delta_j^{(i)}) > -1$,

$|\arg z| < \frac{T\pi}{2}$, $|\arg(z_i)| < \frac{T_i\pi}{2}$, $T > 0$, $T_i > 0$,

$i = 1, \ldots, r, j = 1, \ldots, u^{(i)}$, $L = 1, \ldots, M$, and

m is an arbitrary positive integer5 and the coefficients $A_{n,\alpha'}$ $(n, \alpha' >= 0)$ are arbitrary constant, real or complex.

$$\int_{-1}^{1} \frac{1}{\rho}\,(1+t+\rho)^{-\alpha}\,(1+t+\rho)^{-\beta}\,(1-x)^{\alpha}\,(1+x)^{v}\, S_n^m\left[(1+x)^h\right]$$

$$. H^{M,N}_{P,Q}\left[z\,(1+x)^{h'} \,\middle|\, \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix} \right] H \begin{pmatrix} z_1\,(1+x)^{h_1} \\ z_r\,(1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{v+h'\eta_G+h\alpha'-\beta+1} \sum_{\alpha'=0}^{[n/m]} \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{R=0}^{\infty} \frac{(-n)_{m\alpha'}}{\alpha'\,!}\, A_{n,\alpha'}$$

$$\cdot \frac{(-1)^G \phi(\eta_G)}{G!\,F_g}\; z^{\eta_G}\, \Gamma(1+\alpha+R)\, t^R$$

$$\cdot H^{0,\ \lambda+2\ :\ (u',v');\ \ldots;\ (u^{(r)},v^{(r)})}_{A+2,\ C+2\ :\ [B',D'];\ \ldots;\ [B^{(r)},D^{(r)}]} \left(\begin{matrix} [-v-h'\eta_G-h\alpha';\ h_1;\ \ldots;\ h_r], \\ [(c):\psi',\ \ldots,\ \psi^{(r)}], \end{matrix} \right.$$

$$\begin{matrix} [\beta-v-h'\eta_G-h\alpha':\ h_1;\ \ldots;\ h_r],\ [(a):e';\ \ldots;\ e^{(r)}]: & [(b'):\phi' \\ [\beta-h'\eta_G-h\alpha'+R:h_1;\ \ldots;\ h_r],\ [-\alpha-v-h'\eta_G-h\alpha'-R-1:h_1;\ \ldots;\ h_r]: & [(d'):\delta' \end{matrix}$$

... (17)

where

$\mathrm{Re}(\alpha) > -1$. $\mathrm{Re}(v) > -1$, $h > 0$, $h' > 0$, $h_i > 0$

$$\mathrm{Re}\left(v + h' f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)}\right) > -1,$$

$$|\arg z| < \frac{T\pi}{2},\ |\arg(z_i)| < \frac{T_i\pi}{2},\ T > 0,\ T_i > 0,$$

$i = 1, \ldots, r$, $j = 1, \ldots, u^{(i)}$, $L = 1, \ldots, M$

and m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}$ $(n, \alpha' \geq 0)$ are arbitrary constant, real or complex.

$$\int_{-1}^{1} (1-x)^u (1+x)^v P_w^{(\alpha,\beta)}(t+\rho)\, P_w^{(\alpha,\beta)}(t-\rho)\; S_{n_m}\left[(1-x)^h (1+x)^k\right].$$

$$\cdot H^{M,N}_{P,Q}\left[z\,(1+x)^{h'}\,(1+x)^{k'} \left| \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix} \right. \right] H \left(\begin{matrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{matrix} \right) dx$$

$$= 2^{u+v+(h'+k')\eta_G+(h+k)\alpha'+1}\; \frac{(-1)^w\, \Gamma(1+\alpha+w)\, \Gamma(1+\beta+w)}{(w!)^2}$$

$$\cdot \sum_{\alpha'=0}^{[n/m]} \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{l=0}^{R} \sum_{R=0}^{\infty} \frac{(-n)_{m\alpha'}}{\alpha'!}\, A_{n,\alpha'}\, \frac{(-1)^G \phi(\eta_G)}{G!\,F_g}\, z^{\eta_G}$$

$$\frac{(-R)_l\, (1+\alpha+\beta+R)_l}{(\alpha+1)_l\, l!}\; \frac{(-W)_R\, (1+\alpha+\beta+w)_R}{\Gamma(1+\alpha+R)\, \Gamma(1+\beta+R)}\, t^R$$

$$\cdot H^{0,\ \lambda+2\ :\ (u',v');\ \ldots,\ (u^{(r)},v^{(r)})}_{A+2,\ C+1\ :\ [B',D'],\ \ldots\ [B^{(r)},D^{(r)}]} \left(\begin{matrix} [-v-h'\eta_G-k\alpha':k_1;\ \ldots;\ k_r], \\ [(c):\psi';\ \ldots;\ \psi^{(r)}], \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ \ldots;\ [(b^{(r)}):\phi^{(r)});\ 2^{h_1+k_1} z_1 \\ [(d'):\delta'];\ \ldots;\ [(d^{(r)}):\delta^{(r)}];\ 2^{h_r+k_r} z_r \end{matrix} \right), \qquad \ldots (18)$$

where

$\mathrm{Re}(u) > -1, \mathrm{Re}(v) > -1, h > 0, k > 0, h' > 0, k' > 0, h_i > 0, k_i > 0,$

$\mathrm{Re}(u + h' f_L/F_L + \sum_{i=1}^{r} h_i \, d_j^{(i)}/\delta_j^{(i)}) > -1,$

$\mathrm{Re}(u + h' f_L/F_L + \sum_{i=1}^{r} k_i \, d_j^{(i)}/\delta_j^{(i)}) > -1,$

$| \arg z | < \dfrac{T\pi}{2}, \ | \arg (z_i) | < T_i \pi/2, T > 0, T_i > 0,$

$i = 1, \ldots, r, \ j = 1, \ldots, u^{(i)}, L = 1, \ldots, M$, and
m is an arbitrary positive integer and the coefficients A_n, $(n, \alpha' \geq 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} (1-x)^u (1+x)^v \rho^w P_w^{\prime}(\alpha, \alpha) \left(\frac{1-xt}{\rho}\right) S_n^m \left[(1-x)^h (1+x)^k\right]$$

$$. H_{P,Q}^{M,N}\left[z(1-x)^{h'} (1+x)^{k'} \left| \begin{matrix} e_1, E_P) \\ (f_Q, F_Q) \end{matrix}\right.\right] H \begin{pmatrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{pmatrix} dx$$

$$= 2^{u+v+(h'+k')\eta_G + (h+k)\alpha' + 1} \frac{\Gamma(1+\alpha+w)}{w!}$$

$$. \sum_{\alpha'=0}^{[n/m]} \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{l=0}^{R} \sum_{R=0}^{w} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} \frac{(-1)^G \phi(\eta_G)}{G! F_g} z^{\eta_G}$$

$$. \frac{(-R)_l (2\alpha+R+1)_l}{(\alpha+1)_l \, l!} \frac{(-w)_R}{\Gamma(1+\alpha+R)} t^R$$

$$. H_{A+2, C+1 : [B', D']; \ldots; [B^{(r)}, D^{(r)}]}^{0, \ \lambda+1 \ : (u'+v'); \ldots; (u^{(r)}, v^{(r)})} \left(\begin{matrix} [-v - h'\eta_G - k\alpha' : k_1; \ldots; k_r], \\ [(c) : \psi'; \ldots; \psi^{(r)}] \end{matrix} \right.$$

$$[-u - h\alpha' - l : h_1; \ldots; h_r], [(a) : e'; \ldots; e^{(r)}] :$$
$$[-u - v - (h'+k')\eta_G - (h+k)\alpha' - l - 1 : h_1 + k_1; \ldots; h_r + k_r] :$$

$$\left. \begin{matrix} [(b') : \phi']; \ldots; [(b)^{(r)}) : \phi^{(r)}] \ ; \ 2^{h_1+k_1} z_1 \\ [(d') : \delta']; \ldots; [(d^{(r)}) : \delta^{(r)}] \ ; \ 2^{h_2+k_r} z_r \end{matrix} \right), \qquad \ldots (19)$$

where

$Re(u) > -1, Re(v) > -1, h > 0, k > 0, h' > 0, k' > 0, h_i > 0, k_i > 0,$

$$Re\left(u + h' f_L/F_L + \sum_{i=1}^{r} h_i \, d_j^{(i)} / \delta_j^{(i)}\right) > -1,$$

$$Re\left(v + k' f_L/F_L + \sum_{i=1}^{r} k_i \, d_j^{(i)}/\delta_j^{(i)}\right) > -1,$$

$| arg \, z | < \dfrac{T\pi}{2}, \ | arg \, (z_i) | < \dfrac{T_i\pi}{2}, T > 0, T_i > 0,$

$i = 1, \ldots, r,\ j = 1, \ldots, u^{(i)},\ L = 1, \ldots, M$

and m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}$ $(n, \alpha' \geq 0)$ are arbitrary constants, real or complex.

$$\int_{-1}^{1} (1-x)^{u} (1+x)^{v} \frac{1}{\rho} (1-t+\rho)^{-\alpha} (1+t+\rho)^{-\beta}$$

$$. S_n^m \left[1-x)^h (1+x)^k\right] H_{P,Q}^{M,N}\left[z (1-x)^{h'} (1+x)^{k'} \,\middle|\, \begin{matrix} (e_p, E_p) \\ (f_Q, F_Q) \end{matrix}\right]$$

$$. H \begin{pmatrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ \vdots \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{pmatrix} dx$$

$$= 2^{-\alpha-\beta+u+v+(h'+k')\eta_G+(h+k)\alpha'+1}$$

$$. \sum_{\alpha'=0}^{[n/m]} \sum_{g=1}^{M} \sum_{G=0}^{\infty} \sum_{R=0}^{R} \sum_{R=0}^{\infty} \frac{(-n)_{m\alpha'}}{\alpha'!} A_{n,\alpha'} \frac{(-1)^G \phi(\eta_G)}{G!\,F_g} z^{\eta_G}$$

$$. \frac{(-R)_l\, (1+\alpha+\beta+R)_l}{(\alpha+1)_l\; l!}\, t^R$$

$$. H_{A+2,\, C+1 : [B', D'] ; \ldots ; [B^{(r)}, D^{(r)}]}^{0,\ \lambda+2\ : (u', v') ; \ldots ; (u^{(r)}, v^{(r)})} \left(\begin{matrix} [-v-h'\eta_G-k\alpha' : k_1 ; \ldots ; k_r], \\ [(c) : \psi' ; \ldots ; \psi^{(r)}], \end{matrix} \right.$$

$$[-u-h\alpha'-l : h_1 ; \ldots ; h_r],\ [(a) : \theta' ; \ldots ; \theta^{(r)}] :$$
$$[-u-v(h'+k')\eta_G-(h+k)\alpha'-l-1 : h_1+k_1 ; \ldots ; h_r+k_r] :$$

$$\left. \begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; 2^{h_1+k_1} z_1 \\ \vdots \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; 2^{h_r+k_r} z_r \end{matrix} \right), \qquad \ldots (20)$$

where

$Re(u) > -1,\ Re(v) > -1,\ h > 0,\ k > 0,\ h' > 0,\ k' > 0,\ h_i > 0,\ k_i > 0,$

$$Re\left(u + h' f_L/F_L + \sum_{i=1}^{r} h_i\, d_j^{(i)}/\delta_j^{(i)}\right) > -1,$$

$$Re\left(v + k' f_L/F_L + \sum_{i=1}^{r} k_i\, d_j^{(i)}/\delta_j^{(i)}\right) > -1,$$

$|\arg z| < T\pi/2,\ |\arg(z_i)| < T_i\pi/2,\ T > 0,.\ T_i > 0,$

$i = 1, \ldots, r,\ j = 1, \ldots, u^{(i)},\ L = 1, \ldots, M,$ and
m is an arbitrary positive integer and the coefficients $A_{n,\alpha'}$ $(n, \alpha' \geq 0)$ are arbitrary constants, real or complex.

Proof : To establish (15), multiply both the sides of (10) by

$$(1-x)^{\alpha}(1+x)^{\upsilon} S_n^m\ [(1+x)^h]\ H_{P,Q}^{M,N}\left[z(1+x)^{h'}\ \middle|\ \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right] . H\begin{pmatrix} z_1(1+x)^{h_1}\\ \vdots \\ z_r(1+x)^{h_r}\end{pmatrix}$$

and integration both sides with respect to x between the limits - 1 to 1, we get

$$\int_{-1}^{1}(1-x)^{\alpha}\ (1+x)^{\upsilon} P_w^{(\alpha,\beta)}(t+\rho)\, P_w^{(\alpha,\beta)}(t-\rho)\ S_n^m\left[(1+x)^h\right]$$

$$. H_{P,Q}^{M,N}\left[z(1+x)^{h'}\ \middle|\ \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right] H\begin{pmatrix} z_1(1+x)^{h_1}\\ \vdots \\ z_r(1+x)^{h_r}\end{pmatrix}\qquad dx$$

$$=\int_{-1}^{1}\frac{(-1)^w(1+\alpha)_w(1+\beta)_w}{(w\,!)^2}\quad\sum_{R=0}^{w}\frac{(-w)_R(1+\alpha+\beta+w)_R}{(1+\alpha)_R(1+\beta)_R}$$

$$. P_R^{(\alpha,\alpha)}(x)\ t^R(1-x)^{\alpha}\ (1+x)^{\nu}\ S_n^m\left[(1+x)^h\right] H_{P,Q}^{M,N}\left[z(1+x)^{h'}\ \middle|\ \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]$$

$$. H\begin{pmatrix} z_1(1+x)^{h_1}\\ \vdots \\ z_r(1+x)^{h_r}\end{pmatrix}\qquad dx\,, \qquad \text{... (21)}$$

Now, express the $H_{P,Q}^{M,N}\left[z(1+x)^{h'}\ \middle|\ \begin{matrix}(e_p, E_p)\\ (f_Q, F_Q)\end{matrix}\right]$ functions on the right hand side of (21) by its series (7), then interchange the order of integration and summation in th right hand side of (21), which is justified due to the sbsolute convergent of the integral involved in the process and then evaluate the inner integral with the help of (13).

Finally, interpreting by virture of (1), we get the desired result.

The proofs of the formulas (16) to (20) can be developed by prceeding on similar lines with the help of the result (11) through (14) respectively.

3. SPECIAL CASES

Letting $h' \to 0$, the results in (13) to (20) reduces to known results recently obtained by Chaurasia and Girdhari Lal ([2], *eqn*.(1.11), *p*.99, *eqn*. (1.12), *p*.100 and *eqn*. (2.1) to (2.6), *pp*. 101-104) respectively.

The importance of our resluts lies in its manifold generality. In view of the generality of the polynomials $S_n^m[x]$ on suitbly specializing the coefficients $A_{n,\alpha}$, and making a free use of the special cases of $S_n^m[x]$ listed by Srivastava and Singh [10], our results can be reduced to a large number of integrals involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials, and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in Fox's H-function and in the multivaribale H-function from our results, several integrals involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H functions of one and several variables. Thus the results presented in this paper would at once yield a very large number of integrals, involving a large variety of polynomials and various special functions occurring in the problems, of mathematical analysis.

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REFERENCES

[1] F. Brafman, Generating function of Jacobi and related polynomials, *Proc.Amer. Math.Soc* **.2** (1951), 942-949.

[2] V.B.L. Chaurasia and Girdhari Lal, Some finite integrals involving the product of the multivariable H-function and a general class of polynomials, Jn̄ān̄bha **21** (1991), 97-106.

[3] A.erdelyi et al., *Higher Transcendental Fucntions*, Vol.2, McGraw-Hill, New York, 1953.

[4] A.erdelyi et al., Tables of Integral Transforms, Vol.2, McGraw-Hill, New York, 1954.

[5] C. Fox, The G and H functions as symmetrical Fourier Kernels, *Trans. Amer. Math. Soc.* **98** (1961), 395-429.

[6] E.D. Rainville, *Special Functions, Macmillan*, New York, 1960.

[7] H.M. Srivastava and R. Panda, Expansion theorems for the H function of several complex variables. J. Reine Angew. Math. 288 (1976), 129-145.

[8] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Funtions of One and Two Variables with Applications,* South Asian Publishers, New Delhi and Madras, 1982.

[9] H.M. Srivastava, A contour integral involving Fox's H- function, *Indian J. Math* **14** (1972), 1-6.

[10] H.M. Srivastava and N.P. Singh, The integration of crtain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* (2) **32** (1983), 157-187.

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(Dedicated to Professor S.A.Naimpally on his 62th Birthday)

INEQUALITIES INVOLVING *H*-FUNCTIONS

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ABSTRACT

Six inequalities involving Fox's *H*-functions are established here with the help of certain known inequalities involving ${}_2F_1$ and ${}_3F_2$. Four inequalities involving Fox's *H*-function and ${}_2F_1$ are obtained as particular cases of the results established.

1. INTRODUCTION

Koti [2] established a number of inequalities involving the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ with the help of certain statistical techniques. The aim of this paper is to establish six inequalities involving Fox's H-function by employing thee inequalites established by Koti [2] and following the approach adopted in Raina and Koul [3]. As a particular cases of the inequalities established here, four inequalites involving Fox's *H*-functions and ${}_2F_1$ are obtained.

The Fox's *H*-function is defined and represented in the following manner :

$$H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \theta(s)\, x^s\, ds\,,\ \omega = (-1)^{1/2} \qquad \text{... (1.1)}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}\,.$$

For further details, existence and convergence conditions of $H^{m,n}_{p,q}$ [.], we refer to Srivastava et al. [4, *pp*.10-13].

2. RESULTS REQUIRED

(a) The following inequalities due to Koti [2,*pp*.391-395, theorem (2.1) and equations (3.5), (4.5)] ar rquired in the sequel :

Let k, m, n and λ be positive intgers and let k' be an odd positive integer. Then for $\alpha, \beta > 0$ and $0 < x < 1$:

$$\sum_{j=0}^{k} \binom{2k}{rj} \frac{(1-x)^{-2j}}{(\beta+2k-2j)} \, {}_2F_1 \left(\begin{matrix} \beta+2k-2j, & 1-\alpha-2j \, ; \\ 1+\beta+2k-2j & ; \end{matrix} \; x \right)$$

$$\geq \sum_{j=0}^{k-1} \binom{2k}{2j+1} \frac{(1-x)^{-2j-1}}{(\beta+2k-2j-1} \, {}_2F_1 \left(\begin{matrix} \beta+2k-2j-1, -\alpha-2j \; ; \\ \beta+2k-2j \end{matrix} \; ; x \right); \qquad \text{... (2.1)}$$

$$\sum_{j=0}^{[k'/2]} \binom{2k}{2j+1} \frac{1(1-x)^{-2j-1}}{(\beta+k'-2j-2j-1)} \, {}_2F_1 \left(\begin{matrix} \beta+2k-1, \; -\alpha-2j \, ; \\ \beta+k'-2j \qquad ; \end{matrix} \; x \right)$$

$$\geq \sum_{j=0}^{[k'/2]} \binom{k'}{2j} \frac{(1-x)^{-2j}}{(\beta+k'-2j)} \, {}_2F_1 \left(\begin{matrix} \beta+k'-2j, \; 1-\alpha-2j \; ; \\ 1+\beta+k'-2j \qquad ; \end{matrix} \; x \right); \qquad \text{... (2.2)}$$

$$\sum_{j=1}^{k} \frac{(-2k)_{2j}}{(2j)\,!} \frac{(1-x)^{2j}}{(\beta+2j)} \, {}_2F_1 \left(\begin{matrix} \beta+2j, \quad 1-\alpha-m+2j \; ; \\ 1+\beta+2j \qquad ; \end{matrix} \; x \right)$$

$$\geq \sum_{j=0}^{k-1} \frac{(-2k)_{2j+1}}{(2j+1!} \frac{(1-x)^{2j+1}}{(\beta+2j+1)} \, 2F_1 \left(\begin{matrix} \beta+2j, 2-2-m+2j' \\ 2+\beta+2j \quad ; x \end{matrix} \right) \qquad \text{... (2.3)}$$

$$\sum_{j=1}^{[k'/2]} \frac{(-k')_{2j}}{(2j)\,!} \frac{(1-x)^{2j}}{(\beta+2j)} \, {}_2F_1 \left(\begin{matrix} \beta+2j, \quad 1-\alpha-m+2j \; ; \\ 1+\beta+2j \qquad ; \end{matrix} \; x \right)$$

$$\geq \sum_{J=0}^{[k\,1/2]} \frac{(-k')_{2j+1}}{(2j+1)\,!} \frac{(1-x)^{2j+1}}{(\beta+2j+1)} \, {}_2F_1 \left(\begin{matrix} \beta+2j+1, \quad 2-\alpha-m+2j \; ; \\ 2+\beta+2j \qquad ; \end{matrix} \; x \right) \qquad \text{... (2.4)}$$

$$\sum_{j=0}^{k} \binom{2k}{2j} \frac{x^{-2j-1}}{(\gamma-n-2j)_\lambda} \, {}_3F_2 \left(\begin{matrix} -n, \quad \gamma-\alpha, \quad \gamma-2j-n \, ; \\ \gamma-2k-n, \gamma+\lambda-2j-n \, ; \end{matrix} \; x \right)$$

$$\geq \sum_{j=0}^{k-1} \binom{2k}{2j+1} \frac{x^{-2j-2}}{(\gamma-n-2j-1)_\lambda} \, {}_3F_2 \left(\begin{matrix} -n, \; \gamma-\alpha, \quad \gamma-2j-1-n \; ; \\ \gamma-2k-n, \gamma+\lambda-2j-1-n \; ; \end{matrix} \; x \right)$$

$$\gamma > \alpha > n+2k \, ; \qquad \text{... (2.5)}$$

$$\sum_{j=0}^{[k'/2]} \binom{k'}{2j+1} \frac{x^{-2j-2}}{(\gamma-n-2j-1)_\lambda} \, {}_3F_2 \left(\begin{matrix} -n, \; \gamma-\alpha, \quad \gamma-2j-1-n \; ; \\ \gamma-k'-n, \gamma+\lambda-2j-1-n \; ; \end{matrix} \; x \right)$$

$$\geq \sum_{j=0}^{[k'/2]} \binom{k'}{2j} \frac{x^{-2j-1}}{(\gamma-n-2j)_\lambda} \, {}_3F_2\left(\begin{matrix} -n, & \gamma-\alpha, & \gamma-2j-n\,; \\ \gamma-k'-n, & \gamma+\lambda-2j-n & ; \end{matrix}\; x\right),$$

$\gamma > \alpha > n + k'$. ... (2.6)

The rsults in (2.5) and (2.6) are obtained from Koti [2,*p*.395, *eqn*. (4.5)] by choosing α_2, β_2 suitably.

(b) The following integrals are required here :

Gupta and Olkha [1,p.207, eqs. (2.1), (2.2)] :

For $\mu, \nu > 0$, $Re(\rho) > 0$

$$\int_0^1 x^{\rho-1} (1-x)^{\alpha-\rho-1} \, {}_2F_1\left(\begin{matrix}\alpha, \beta\,; \\ \gamma\,;\end{matrix}\; x\right)\left[1+z\left(\frac{x}{1-x}\right)^{\mu}\right]^{-\nu} dx$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\nu)\,\Gamma(\alpha)\,\Gamma(\gamma-\beta)} \, H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu),\,(1-\nu,1),\,(\gamma-\rho,\mu) \\ (0,1),\,(\alpha-\rho,\mu),\,(\gamma-\beta-\rho,\mu) \end{matrix}\right.\right],$$

$Re(\alpha + \mu\nu) > \mathrm{Re}(\rho)$; ... (2.7)

$$\int_0^1 x^{\rho-1} (1-x)^{\sigma-1} \, {}_3F_2\left(\begin{matrix} -n, c_2, c_3\,; \\ d_1, d_2\,; \end{matrix}\; x\right)\left(1+zx^{\mu}\right)^{-\nu} dx$$

$$= \frac{\Gamma(\sigma)}{\Gamma(\nu)} \sum_{r=0}^{n} \frac{(-n)_r\,(c_2)_r\,(c_3)_r}{(d_1)_r\,(d_2)_r\; r\,!} \, H_{2,2}^{1,2}\left[z \left| \begin{matrix} (1-\rho-r,\mu),\,(1-\nu,1) \\ (0,1),\,(1-\rho-\sigma-r,\mu) \end{matrix}\right.\right]$$

$Re\ (\sigma) > 0$. ... (2.8)

3. THE MAIN INEQUALITIES

For $\alpha, \beta, \mu, \nu, \sigma, \rho, z > 0$; k, m, n and λ being positive integers and k' an odd positive integer, we have :

$$\sum_{j=0}^{k} \binom{2k}{2j} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu),\,(1-\nu,1),\,(1+\beta+2k-2j-\rho,\mu \\ (0,1),\,(\beta+2k-2j-\rho,\mu),\,(\alpha+\beta+2k-\rho,\mu) \end{matrix}\right.\right]$$

$$\geq \sum_{j=0}^{k-1} \binom{2k}{2j+1} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu),\,(1-\nu,1),\,(\beta+2k-2j-\rho,\mu) \\ (0,1),\,(\beta+2k-2j-\rho-1,\mu),\,(\alpha+\beta+2k-\rho,\mu) \end{matrix}\right.\right] \tag{3.1}$$

$$\sum_{j=0}^{[k'/2]} \binom{k'}{2j+1} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu),\quad (1-\nu,1),\quad (\beta+k'-2j-\rho,\mu) \\ (0,1),\,(\beta+k'-2j-\rho-1,\mu),\,(\alpha+\beta+k'-\rho,\mu) \end{matrix}\right.\right]$$

$$\geq \sum_{j=0}^{[k'/2]} \binom{k'}{2j} H_{3,3}^{3,2}\left[z \left| \begin{matrix} (1-\rho,\mu),\,(1-\nu,1),\,(1+\beta+k'-2j-\rho,\mu) \\ (0,1),\,(\beta+k'-2j-\rho,\mu),\,(\alpha+\beta+k'-\rho,\mu) \end{matrix}\right.\right];$$

... (3.2)

$$\sum_{j=1}^{k} \frac{(-2k)_{2j}}{(2j)\,!}\; H_{3,3}^{3,2}\left[z \,\middle|\, \begin{matrix} (1-\rho,\mu),\,(1-\nu,1),\,(1+\beta+2j-\rho,\mu) \\ (0,1),\,(\beta+2j-\rho,\mu),\,(\alpha+\beta+m-\rho,\mu) \end{matrix} \right]$$

$$\geq \sum_{j=0}^{k-1} \frac{(-2k)_{2j+1}}{(2j+1)\,!}\; H_{3,3}^{3,2}\left[z \,\middle|\, \begin{matrix} (1-\rho,\mu),\quad (1-\nu,1),\quad (2+\beta+2j-\rho,\quad \mu) \\ (0,1),\,(1+\beta+2j-\rho,\mu),\,(\alpha+\beta+m-\rho,\mu) \end{matrix} \right]; \qquad \ldots (3.3)$$

$$\sum_{j=1}^{[k'/2]} \frac{(-k')_{2j}}{(2j)\,!}\; H_{3,3}^{3,2}\left[z \,\middle|\, \begin{matrix} (1-\rho,\mu)\,,\,(1-\nu,1),\,(1+\beta+2j-\rho,\mu) \\ (0,1),\,(\beta+2j-\rho,\mu),\,(\alpha+\beta+m-\rho,\mu) \end{matrix} \right]$$

$$\geq \sum_{j=0}^{[k'/2]} \frac{(-k')_{2j+1}}{(2j+1)\,!}\; H_{3,3}^{3,2}\left[z \,\middle|\, \begin{matrix} (1-\rho,\mu),\quad (1-\nu,1),\quad (2+\beta+2j-\rho,\mu) \\ (0,1),\,(1+\beta+2j-\rho,\mu)\,,\,(\alpha+\beta+m-\rho,\mu) \end{matrix} \right] \qquad \ldots (3.4)$$

$$\sum_{j=0}^{k} \sum_{r=0}^{n} \binom{2k}{2j} \frac{(-n)_r\,(\gamma-\alpha)_r}{r\,!\,(\gamma-2k-n)_r\,(\gamma-n-2j+r)_\lambda}\,.$$

$$\cdot\, H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix} (1-\gamma-\lambda+n+2j-r,\mu)\,,\,(1-\gamma,1) \\ (0,1),\quad (1-\nu+n+2j-\sigma-r,\mu) \end{matrix} \right]$$

$$\geq \sum_{j=0}^{k-1} \sum_{r=0}^{n} \binom{2k}{2j+1} \frac{(-n)_r\,(\gamma-\alpha)_r}{r\,!\,(\gamma-2k-n)_r\,(\gamma-n-2j-1+r)_\lambda}\,.$$

$$H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix} (2-\gamma-\lambda+n+2j-r,\mu),\,(1-\nu,1) \\ (0,1),\,(2-\gamma-\lambda+n+2j-\sigma-r,\mu) \end{matrix} \right],$$

$$\gamma > \alpha > n+2k\,; \qquad \ldots (3.5)$$

$$\sum_{j=0}^{[k',/2]} \sum_{r=0}^{n} \binom{k'}{2j+1} \frac{(-n)_r\,(\gamma-\alpha)_r}{r\,!\,(r-k'-n)_r\,\;(\gamma-n-2j+r-1)_\lambda}\,.$$

$$H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix} (2-\gamma-\lambda+n+2j-r,\mu),\,(1-\nu,1) \\ (0,1),\,(2-\gamma-\lambda+n+2j-\sigma-r,\mu) \end{matrix} \right],$$

$$\geq \sum_{j=0}^{[k'/2]} \sum_{\nu=0}^{n} \binom{k'}{2j} \frac{(-n)_n\,(r-\alpha)_r}{r!\,(r-k-n)_r\,(r(r-n-2j+r)_\lambda}$$

$$H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix} (1-r-\lambda+n+2j-r,\mu),\,(1-nu,1) \\ (0,1),\,(1-r-\lambda+n+2j-\sigma-r,\mu \end{matrix} \right],$$

$$\gamma > \alpha > n + k' . \qquad \dots (3.6)$$

Outline of proofs. To establish the inquality in (3.1), we first multiply both sides of (2.1) by $x^{\rho - 1} (1 - x)^{\beta + 2k - \rho - 1} \left[1 + z \left(\frac{x}{1 - x} \right)^{\mu} \right]^{-\nu}$, integrat with respect to x betwen limits 0 to 1 and then use (2.7) to arrive at the desired result in (3.1). The inequality in (3.2) is obtained by multiplying both sides of (2.2) by $x^{\rho - 1} (1 - x)^{\beta + k' - \rho - 1} \left[1 + z \left(\frac{x}{1 - x} \right)^{\mu} \right]^{-\nu}$ instead and proceeding as above and using (2.7) therein.

The inequalities in (3.3), (3.4), (3,5) and (3.6) are established in a similar manner. We use the multiplying factors

$x^{\rho - 1} (1 - x)^{\beta - \rho - 1} \left[1 + z \left(\frac{x}{1 - x} \right)^{\mu} \right]^{\nu}$ in (2.3) and (2.4) for (3.3) and (3.4) and $x^{\gamma + \lambda - n} (1 - x)^{\sigma - 1} (1 + z x^{\mu})^{-\nu}$ in (2.5), (2.6) for (3.5) , (3.6)) and make use of (2.7) and (2.8) respectively therein.

4. PARTICULAR CASES

1. In view of the formula

$$\underset{\nu \to \infty}{Lt} \left(1 + \frac{x}{\nu} \right)^{-\nu} = e^{-x} ,$$

the inequalities in (3.1) and (3.3) reduce respectively to the following :

For $\alpha, \beta, \mu, \sigma, \rho, z > 0$; $k.$, m and λ being positive integers, we have

$$\sum_{j=0}^{k} \binom{2k}{2j} H_{2,3}^{3,1} \left[z \,\middle|\, \begin{matrix} (1 - \rho, \mu), (1 + \beta + 2k - 2j - \rho, \mu) \\ (0, 1), (\beta + 2k - 2j - \rho, \mu), (\alpha + \beta + 2k - \rho, \mu) \end{matrix} \right] .$$

$$\geq \sum_{j=0}^{k-1} \binom{2k}{2j+1} H_{2,3}^{3,1} \left[z \,\middle|\, \begin{matrix} (1 - \rho , \mu) , (\beta + 2k - 2j - \rho, \mu) \\ (0, 1), (\beta + 2k - 2j - \rho - 1, \mu), (\alpha + \beta + 2k - \rho, \mu) \end{matrix} \right] ;$$

(4.1)

$$\sum_{j=1}^{k} \frac{(-2k)_{2j}}{(2j)\,!} H_{2,3}^{3,1} \left[z \,\middle|\, \begin{matrix} (1 - \rho, \mu), (1 + \beta + 2j - \rho, \mu) \\ (0, 1), (\beta + 2j - \rho, \mu), (\alpha + \beta + m - \rho, \mu) \end{matrix} \right] .$$

$$\geq \sum_{j=0}^{k-1} \frac{(-2k)_{2j+1}}{(2j+1)!} H_{2,3}^{3,2} \left[z \,\middle|\, \begin{matrix} 1 - \rho, \mu), (i + \beta + 2j - \rho, \mu) \\ (0, 1), (1 + \beta + 2j - \rho, \mu), (\alpha + \beta + m - p, \mu) \end{matrix} \right] . \qquad \dots (4.2)$$

2. If, in (3.5) and (3.6), we take $\mu = 1 = k'$, these yield respectively the following inequalities involving ${}_2F_1$ in view of the known relation Srivastava et al. [4,*p*.18, *eqn*. (2.6.3)] :

For k,n, σ and λ being positive integers and $\nu, z > 0$, we have :

$$\sum_{j=0}^{k} \sum_{r=0}^{n} \binom{2k}{2j} \frac{(-n)_r\,(\gamma-\alpha)_r}{r\,!\,(\gamma-2k-n)_r\,(\gamma-n-2j+r)_{\lambda+\sigma}}\; {}_2F_1\left(\begin{matrix} r+\lambda-n-2j+2,\ \nu\ ; \\ \gamma+\lambda+\sigma-n-2j+r\ ; \end{matrix}\ -z\right)$$

$$\geq \sum_{j=0}^{k-1} \sum_{r=0}^{n} \binom{2k}{2j+1} \frac{(-n)_r\,(\gamma-\alpha)_r}{r\,!\,(\gamma-2k-n)_r\,(\gamma-n-2j+r-1)_{\lambda+\sigma}} \cdot {}_2F_1\left(\begin{matrix} \gamma+\lambda-n-2j+r-1,\ \nu\ ; \\ \gamma+\lambda+\sigma-n-2j+r-1\ ; \end{matrix}\ -z\right).$$

$\gamma > \alpha > n+2k$; ... (4.3)

$$\sum_{r=0}^{n} \frac{(-n)_r\,(\gamma-\alpha)_r}{r\,!\,(\gamma-n-1)_{r+\lambda+\sigma}}\; {}_2F_1\left(\begin{matrix} r+\lambda-n+r-1,\ \nu\ ; \\ \gamma+\lambda+\sigma-n+r-1\ ; \end{matrix}\ -z\right)$$

$$\geq \sum_{r=0}^{n} \frac{(-n)_r\,(\gamma-\alpha)_r}{r\,!\,(\gamma-n-1)_r\,(\gamma-n+r)_{\lambda+\sigma}}\; {}_2F_1\left(\begin{matrix} \gamma+\lambda-n+r,\ \nu\ ; \\ \gamma+\lambda+\sigma-n+r\ ; \end{matrix}\ -z\right),$$

$\gamma > \alpha > n+1$. ... (4.4)

The inequalities in (3.1) to (3.6) and (4.3), (4.4) are also capable of yielding inequalities involving similar H-functions, ${}_1F_1$, ${}_2F_2$ and ${}_3F_2$, but these are not recorded here for lack of space.

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RFERENCES

[1] K.C.Gupta and G.S.Olkha, Intgrals involving products of generalizd hypergeometric functions and Fox's H-function, *Univ. Nac. Tucuman Rev. Ser. A* **19** (1969), 205-212.

[2] K.M.Koti, Som inequalities in Hypergeometric functions using statistical techniques, *Indian J.Pure Appl. Math.* **22** (1991), 389- 396.

[3] R.K.Raina and C.L.Koul, Some inequalities involving the Fox's H-function, *Proc. Indian Acad. Sect. A* **83** (1976), 33-40.

[4] H.M.Srivastava, K.C.Gupta and S.P.Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

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NATURE OF EQUILIBRIUM POINTS IN CERTAIN SYMMETRIC MATRICES)

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ABSTRACT

In this paper we apply the fixed point algorithm to investigate th nature of the quilibrium points in the symmetric matrices that occur in convex quadratic programming and in many decision problems. Lmak, Cottle and Dantzig have already shown that every linear programming problem and convex quadratic programming problem can be transformed into a linear complementary method based on fixed point algorithm has been applied hr, and we also discuss the behaviour of equilibrium points for symmetric matrix M of order 3 statisfying the following form of LCP.

$$I_w - M_z - z_0 e = q$$

$$w \geq 0 \, ; z \geq 0 \, ; z_0 \geq 0$$

$$w_i z_i = 0 \, ; i = 1, 2, \ldots, n$$

1. Introduction

To solve the non-linear mathematical problems there is no unique simplex type procedure as is in the cases of linear mathematical problems. Therefore, many non-linear algorithms, known as quadratic programming, separable programming, geometric programming, fractional programming, and dynamic programming, have been developeed according to the nature of the non-linear problem. All these important algorithms that are used to find the solution of non-linear problems aree based on the Khun-Tucker condition. Due to complexity of these algorithms, Lemke [3] and Cottle and Dantzig [1], have shown that every programming problem and convex quadratic problem can be transformed into linear complementary problem. Thus the LCP provides a unifying format for studying a large class of important problems in mathematical programming and in there theory of matrix games. Lemke and Howson [2] developed a complementary pivot method for solving such LCP pertaining to the problem of computing equilibrium points in bimatrix gaem. Prior, to the Lemke methods, Cottle and Dantzig [1] proposed an algorithm known as principle pivoting method to solve *LCP*, but Lemke's algorithm is logically simpler than that of Cottle and Dantzig. Moreover, it can be used to process the wider class of complementary problem. Lemke [3] has also extended his method for solving a general *LCP*. This general method is normally known as complementary pivot method or Lemke's complementary pivot method.

The main characteristics of Lemke's technique for the solution of general form of *LCP* given by

$$Mz - w = q \; ; \; z \geq 0 \; ; w \geq 0, \tag{1.1}$$

where M is any square matrix

It is based on the concept of non-degeneracy and the generation of adjacent extreme point path which terminates in an equilibrium point, if it exists. Geometrically, the system of the form (1.1) can be visualised as convex polyhedron in z-space of points satisfying :

$$Mz \geq q \; ; \; z \geq 0$$

The path of the points generated consist those points for which

$$z^T w = \sum_{i=1}^{n} z_i w_i$$

that is, points for which the sum has at most summand $z_i w_i$ positive for fixed i.

Moreover, a point of (1.1) which reduces $z^T w = 0$ is called an equilibrium point.

In this paper we have considered a complementary problem of the form :

$$Iw - Mz - z_0 e = q$$

$$w \geq 0 \; ; z \geq 0 \; ; z_0 \geq 0,$$

where M is a symmetric 3×3 matrix whose equilibrium points exist.

We have tried to show under what conditions the equilibrium points for symmetric matrix remains unaltered, and also that any new equilibrium point is a multiple of the previously obtained equilibrium point.

2. Notations and Definitions

The General form of the set of LCP in R^n is denoted by :

$$Z = \{ z : w - Mz = q : z \geq 0 \; ; w \geq 0 \} \tag{A}$$

2.1 Definition :

In any *LCP* with M as the data, the pair of column vectors $\{ I.j . - M.j \}$ is called as j^{th}. Complementary pair of column vector for $j = 1,2,\ldots,n$ denoted by $A._j$

The ordered set of vectors $(A._1 \ldots A._n)$ is called as complementary set of vectors.

To determine these complementary set of vectors $(A_{\cdot 1} \dots A_{\cdot n})$; for any given pair (q,m) in *LCP*, the process similar to *LPP* is applied.

3. Preliminary The following result has been proved by Lemke [3] to show the existence and characterisation of equilibrium points.

3.1 Theorem : For fixed s, if Z is non-degenerate, Z_s is either empty or the union of disjoint adjacency paths of Z. Then the set S of equilibrium points of Z is precisely the set of end-points of adjacency paths comprising Z_s.

Based on theorem (3.1) the following lemma and theorem gives the existence of equilibrium points in *LCP*.

3.2 Lemma : Z is empty if there exists a $u > 0$ satisfying :

$$(3.2.1) \qquad M^T u \le 0 \text{ ; and } u^T q > 0.$$

3.3 Theorem : Let Z be non-degenerate and have the property that for some s, Z_s contains precisely one ray of Z. The Z has an odd number of equilibrium points.

The consequence of the above theorem is the following corollary for the bimatrix game.

3.4 Corollary :Let Z be non-degenerate if $q = e$, and $M > 0$ then Z has an odd number of equilibrium points.

The following theorm gives the uniqueness condition of equilibrium points.

3.5 Theorem :If Z is non-degenerate and M satisfies $z^T Mz \ge 0$ (that is, if M is non- negative definite) then there exists at most one equilibrium point.

By similar process the equilibrium points for more wider class of problems of the form :

$$(B) \qquad Z^* = \left\{ z^* : M_Z + z_0 e - w = q : w, z \ge 0 \text{ , } z_0 \ge 0 \right\} \qquad R^{n+1}$$

can be obtained

where $z^* = \begin{pmatrix} z \\ z_0 \end{pmatrix}$. the details can be found in [3].

Since in this paper we mainly concentrated on the computation procedure applied to *LCP* for the determination of equilibrium points when matrix M is symmetric, so the steps of alogrithm that has been used to compute the equilibrium points *LCP* are similar to *LPP* problem except the following :

(i) The *LCP* is taken in the form as :

$$Iw - Mz - z_0 e = q$$

$$; w \ge 0, \; z \ge 0, z_0 \ge 0$$

expressed in tableau form as

w $\quad$ z $\quad$ z_0	
I $\quad$ $-M$ $\quad$ $-e$	q

(*a*) If at some stage the variable z_0 becomes the outgoing variable. At this stage the basic vector at the end if the pivot step is a complementary feasible basic vector for the *LCP* given by (*i*) and thus the basic solution corresponding to it is a solution of the *LCP* (*i*).

(*b*) If at some stage z_0 may still be a basic vector and the incoming variable just the repetition of the previous steps or the pivot column may be non-positive. When this happen the algorighm terminate as it has reached the stage where, it is unable to solve the *LCP* given by (*i*). This type of termination is called ray termination.

4. Main Result. To study the behaviour of symmetric matrix in LCP while determining the equilibrium points, let us first consider following symmetric matrix in *LCP* given below :

Let

4.1 $$Iw - Mz - z_0e = q$$

where

4.2 $$M = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

and

4.3 $$q = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

On applying (4.2) and (4.3) in (4.1) the initial tableau form takes the form as :

(4.4)

w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
1	0	0	-4	-2	-1	-1	-4
0	1	0	-2	-4	-2	$\boxed{-1}$	-6
0	0	1	-1	-2	0	-1	-2

As per *LCP* algorithm the starting pivot point is -1 of second element in z_0 corresponding to least value -6 of q. On applying *LCP* rules so that Z_0 to be the element of basic vector as well as to obtain equilibrium points, the integration step are shown in three subsequents tables as given below :

4.5

Basic variable	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
w_1	1	− 1	0	− 2	②	1	0	2
z_0	0	− 1	0	2	4	2	1	6
w_3	0	− 1	1	1	2	2	0	4
z_2	1/2	− 1/2	0	− 1	1	1/2	0	1
z_0	− 2	1	0	⑥	0	0	1	2
w_3	− 1	0	1	3	0	1	0	2
z_2	1/6	− 1/3	0	0	1	1/2	1/6	4/3
z_1	− 1/3	1/6	0	1	0	0	1/6	1/3
w_3	0	− 1/2	1	0	0	1	− 1/2	1

Thus we have $z = (1/3, 4/3, 0)$ and $w = (0,0,1)$, satisfying $(w)_i, (z)_i = 0(w), (z)_, = 0$. Hence

$$z = (1/3,\ 4/3,\ 0) \tag{4.6}$$

is an equilibrium point.

Again, on replacing the third element of first row of symmetric matrix of (4.2) by another element so that M remains a symmetric matrix, we observe that equilibrium points remains unaltered except value of w changes which can be seen by considering the collowing matrix and the corresponding tableau as shown below :

Let

$$\text{(4.7)}\quad M = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \text{(4.8)}\quad q = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

(4.9)

w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
1	0	0	− 4	− 2	− 3	− 1	− 4
0	1	0	− 2	− 4	− 2	− 1	− 6
0	0	1	− 3	− 2	0	− 1	− 2

By applying integration steps similar to (4.5) we have $z = (1/3, 4/3, 0)$ but

$$w = (0,0,5/3), \text{ yet satisfying } (w)_i\,(z)_i = C$$

thus

$$z = (1/3,\ 4/3,\ 0)\,. \tag{4.10}$$

is again an equilibrium point.

But if we replace third element of the second row of the matrix (4.2) some other element then the equilibrium point does not exist as shown below :

Let

$$M = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 5 \\ 1 & 5 & 0 \end{pmatrix} \tag{4.11}$$

and

$$q = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix} \tag{4.12}$$

we have

(4.13)

w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
1	0	0	− 4	− 2	− 1	− 1	− 4
0	1	0	− 2	− 4	− 5	− 1	− 6
0	0	1	− 1	− 5	0	− 1	− 2

(4.14)

Basic variables	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
w_1	1	− 1	0	− 2	2	④	0	2
z_0	0	− 1	0	2	4	5	1	6
w_3	0	− 1	1	1	− 1	5	0	4
z_3	1/4	− 1/4	0	− 1/2	1/2	1	0	1/2
z_0	− 5/4	1/4	0	9/2	3/2	0	1	7/2
w_3	− 5/4	1/4	1	7/2	− 7/2	0	0	3/2
z_3	− 3/8	− 3/4	2/7	0	0	1	0	10/14
z_0	5/14	− 1/14	− 9/7	0	− 3	0	1	11/7
z_1	− 5/14	1/14	2/7	1	− 1	0	0	3/7
z_3	− 1/56	23/28	0	− 1	1	1	0	2/7
z_0	− 38/28	1/49	0	9/2	− 15/2	0	1	49/14
z_3	− 5/4	1/4	1	7/2	− 7/2	0	0	3/2

(Since there is a repetition of the tableau, so it has no equilibrium point.

Moreover, we have also observed that if we alter any one element of the diagonal then a new equilibrium point is obtained. Besides, if we consider the element of q in (4.3) as its some multiple, then correspondingly equilibrium points also increase by the same multiple.

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REFERENCES

[1] R.W. Cottle and G.B. Dantzig, Complementary theory of mathematical programming, *Linear Algebra Appl.* **1** (1968), 103-125.

[2] C.E. Lemke, Bimatrix equilibrium points and mathematical programming, *Managemen Sci.* **4** (1965), 681-689.

[3] C.E. Lemke and J.T. Howson, Equilibrium points of bimatrix games, *SIAM J. Appl. Math.* **12** (1964), 413-423.

[4] K.G. Murthy, Computational complexity of complementary pivot methods, *Mathematical Programming Study* **7** (1978), 61-73.

[5] D.P. Shukla and N.P.S. Bawa, New triangulation of R^n *Proc. Indian Acad. Sci. Math. Sci.* **100** (1990), 79-85.

[6] J.A. Tomlin, Robust Implementation of Lemke's method for the linear complementary problem, *Mathematical Programming Study* **7** (1978), 55-60.

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A NEW "LOGISTIC" DIFFERENCE EQUATION

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ABSTRACT

The derivation of the usual logistic curve as a model of restricted growth such as the spread of rumour or disease depends on several unwarranted assumptions. In this note we consider the model in more detail an deduce and equation very similar to the usual one but seems to be better hehaved.

If a rumour or a disease is being spread in a population of size 100 by the mechanism "each person meets k other people each day - k being a constant, and spreads the rumour/disease at every opportunity, and no one ever forgets the rumour/recovers from the disease", and z_n is the number of people who know the rumour/have the disease at the beginning of the n-th day, then the difference equation

$$z_{n+1} = z_n + kz_n(100 - z_n)/100 \qquad \text{... (1)}$$

with an initial condition given by a value z_1, where $0 < z_1 < 100$, is commonly used to describe the spread, see [1] for example. (It is usually written in the form $u_{n+1} = au_n(1 - u_n)$, where $a = 1 + k$ and $u_n = kz_n/100(1 + k)$. The continuous analogue, where the spread is continuous in time, has the solution :

$$z(t) = 100/(1 + Ce^{-kt}) \qquad \text{... (2)}$$

(C is a constant determined by z(0)), the so-called "logistic curve".

First we will review how (1) is obtained. We state the argument in terms of rumour. At the beginning of the n-th stage, when z_n people know the rumour, the proportion of ignorant people in the population is $(100 - z_n)/100$, so each of the people meets k people during the subsequent time interval, of whom $k(100 - z_n)/100$ learn the rumour, and (1) results. This argument is obviously dubious : the second order effect - each person who learns the rumour during the n-th stage will pass it on to others during that stage (which increases the number who learn the rumour) - and the interaction of the z_n people with each other - the fact that each of the z_n, by spreading the rumour, decreases the probability for the next person (which decreases the number who learn the rumour) - are both ignored. Also, each of the z_n

people contacts k people out of 99 (ie, not himself) so the ratio should be $(99 - (z_n - 1))/99 = (100 - z_n)/99$, a small difference which does not address the main objections. This change results in this solution to the continuous analogue :

$$z(t) = 100/(1 + Ce^{-100kt/99}) . \qquad \ldots (3)$$

The models (1) describes suggest that $z_n < 100$ for all n, but a sequecne defined by (1) may overshoot 100 - this occurs if and only if $k > 1$, and certainly such a valure of k is not unrealistic. In this note we will show that the difference equation.

$$u_{n+1} = 100 - (100 - u_n)\,(98/99)^{cu_n} \qquad \ldots (4)$$

with $0 < u_1 < 100$ and $c = -k/(100 \ln 98/99) = .985k$ provides a better description which does not overshoot 100. In fact, it is immediately clear that the sequence $\{u_n\}$ increases monotonically to 100, as one would expect, and there are no bifuractions, no chaos. Also, the solution to the continuous case is (2) with a slight change in k (see equation (8) below).

We phrase the argument in terms of rumour. Given u_n, to find u_{n+1} we suppose that the u_n knowledgeable people queue up, then the first person in the queue meets a random selection of k people in the population at large, then the second perosn meets k people, and so on, until each of the u_n people in the queue have met their random selection of k people. The rumour is passed on at every opportunity, but we will not consider "second order rumourmongering", so our value for u_{n+1} will necessarily be an underestimate of the true value.

Consider the first person in the queue. He selects one of 99 people of whom $u_n - 1$ are knowledgeable; thus the person he meets has the probability of $(100 - u_n)/99$ of being ignorant, so the rumour is passed on to $(100 - u_n)/99$ people. For the next person met, the probability of him being ignorant is

$$(100 - (u_n + (100 - u_n)/99))/99$$

and so on. The first person passes the rumour on to

$$(100 - u_n)/99 + (100 - (u_n + (100 - u_n)/99))/99 +$$
$$(1000 - (u_n + (100 - u_n)/99 + (100 - (u_n + (100 - u_n)/99))/99 + \ldots$$

to k terms, and this is $(100 - u_n)\,(1 - (98/99)^k)$.

$$\ldots (5)$$

The second person starts off with

$$u_n + (100 - u_n)(1 - (98/99)^k) \qquad \ldots (6)$$

knowing the rumour, so the number of people who learn it from him is (5) with u_n replaced by (6). Continuing this for all u_n people in the queue, and adding up, we get

$$u_{n+1} = 100 - (100 - u_n)(98/99)^{ku_n}. \qquad ..(7)$$

The differential equation arising form this is easily derived: if the rumour is being continuously spread and u(t) is the number of people who know the rumour at time t, then

$$\begin{aligned} du/dt &= lim\ (u(t+h) - (u(t)/h \\ &= lim\ (100 - u(t)t)(1 - (98/99)^{khu})/h \\ &= (100 - u)\ (-\,ln(98/99))ku \qquad \text{(l'Hospital)} \qquad ...\ (8) \end{aligned}$$

which is just the differential equation for the usual logistic curve, but with k replaced by $-\ 100k\ ln(98/99), = 1.015k$. (It is surprising that (7) doesn't converge to (3) - evidently the second effect and/or the interaction effect do not vanish for short time intervals, but the discrepency is small: 1.0150 is near 100/99.)

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REFERENCE

[1] E. Batschelet, *Introduction to Mathematics for Life Scientists,* Springer-Verlag, New York, Heidelberg, and Berlin, 1973.

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SOME FIVE SERIES EQUATIONS INVOLVING GENERALISED BATEMAN K- FUNCTIONS

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ABSTRACT

Solution of certain five series equations involving generalised Bateman K-functions, has been obtained in this paper.

1. INTRODUCTION

Dual and triple series equations arise in many mixed boundary value problems of mathematical physics [8]. The above equations are the generalisation of dual, triple and quadruple series equations considered earlier by [2] , [3] and [9]. Recently Dwivedi and Pandey [4] have obtained solution of the above set of equations but calculations are erroneous. Therefore the object of this paper is to find the solution of the same set of equations whose solutions appear to be correct. Various other authors [1], [5], [6] and [7] have considered other set of series equations involving different special functions.

2. THE EQUATIONS

We shall solve the following system of five series equations involving generalised Bateman K-functions :

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta+\sigma+n+1)} K_{2\,(n+\alpha)}^{2\,(\alpha+\sigma)}(x) = \begin{cases} f_1(x) , & 0<x<a \\ f_3(x) , & b<x<c \\ f_5(x) , & d<x<\infty \end{cases} \quad \text{... (2.1)}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\nu+\sigma+n+1)} K_{2\,(n+\beta)}^{2\,(\beta+\sigma)}(x) = \begin{cases} f_2(x) , & a<x<b \\ f_4(x) , & c<x<d \end{cases} \quad \text{... (2.2)}$$

where $K_{2\,(n+\alpha)}^{2\,(\alpha+\sigma)}(x)$ is the generalised Bateman K-function, A_n is unknown coefficient, $f_1(x), f_2(x) , f_3(x) , f_4(x)$ and $f_5(x)$ are known functions and the parameters $\alpha, \beta, \nu, \sigma$ all are > -1. It is shown here that the problem of solving five series equations can be reduced to that of solving simultaneous Fredholm integral equations of second kind.

3. SOLUTION OF FIVE SERIES EQUATIONS

Let us suppose

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta+\sigma+n+1)} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = \begin{cases} \phi_1(x), & a<x<b \\ \phi_2(x), & c<x<d \end{cases} \quad \ldots (3.1)$$

Using orthogonality relation, we get from (2.1) and (3.1)

$$A_n = \frac{\Gamma(2\beta+\sigma+n+1)\,\Gamma(2\alpha+\sigma+n+1)}{2^{2\alpha+2\sigma}\,\Gamma(n-\sigma)} \left[\int_0^a f_1(x) + \int_a^b \phi_1(x) + \int_b^c + f_3(x)\right.$$

$$\left. + \int_c^d \phi_2(x) + \int_d^{\infty} f_5(x) \quad x^{-2\alpha-2\sigma-1} K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x)\, dx. \quad \ldots (3.2)$$

Substituting this expression for A_n from (3.2) in (2.2), we have

$$\int_a^b \phi_1(r)\, r^{-2\alpha-2\sigma-1} S(r,x)\, dr + \int_c^d \phi_2(r)\, r^{-2\alpha-2\sigma-1} S(r,x)\, dr$$

$$= \begin{cases} M(x), & a<x<b \quad \ldots (3.3) \\ N(x), & c<x<d \quad \ldots (3.4) \end{cases}$$

where,

$$M(x) = \frac{\{\Gamma(2\nu+\sigma+n+1)\}^2}{\Gamma(2\beta+\sigma+n+1)\,\Gamma(2\alpha+\sigma+n+1)} f_2(x) - \left[\int_0^a f_1(r) + \int_b^c f_3(r)\right.$$

$$\left. + \int_d^{\infty} f_5(r)\right] r^{-2\alpha-2\sigma-1} S(r,x)\, dr \quad \ldots (3.5)$$

$$N(x) = \frac{\{\Gamma(2\nu+\sigma+n+1)\}^2}{\Gamma(2\beta+\sigma+n+1)\,\Gamma(2\alpha+\sigma+n+1)} f_4(x)$$

$$-\left[\int_0^a f_1(r) + \int_b^c f_3(r) + \int_d^{\infty} f_5(r)\right] r^{-2\alpha-2\sigma-1} S(r,x)\, dr \quad \ldots (3.6)$$

and

$$S(r;x) = \sum_{n=0}^{\infty} \frac{\Gamma(2\nu+\sigma+n+1)}{2^{2\beta+2\sigma}\,\Gamma(n-\sigma)} K_{2(n+\beta)}^{2(\beta+\sigma)}(r)\; K_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) \quad \ldots (3.7)$$

$$= \frac{e^{-x}\, 2^{2\alpha-2\nu}}{\Gamma(2\alpha-2\nu)\,\Gamma(2\beta-2\nu)} \int_0^t E(\xi)\,(x-\xi)^{2\alpha-2\nu-1}\,(r-\xi)^{2\beta-2\nu-1}\, d\xi \quad \ldots (3.8)$$

$$= \frac{e^{-x}\, 2^{2\alpha-2\nu}}{\Gamma(2\alpha-2\nu)\,\Gamma(2\beta-2\nu)} S_t(r,x) \quad \ldots (3.9)$$

Starting with equation (3.3) and making some calculations, we derive

$$\int_a^x \frac{E(\xi)\,\Phi_1(\xi)}{(x-\xi)^{1-2\alpha+2\nu}}\, d\xi = \frac{\Gamma(2\alpha-2\nu)\Gamma(2\beta-2\nu)}{2^{2\alpha-2\nu}} e^x M(x) - \int_0^a \frac{E(\xi)}{(x-\xi)^{1-2\alpha+2\nu}}\, d\xi$$

$$\times \int_a^b \frac{r^{-2\alpha-2\sigma-1}\,\phi_1(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr - \int_0^x \frac{E(\xi)}{(x-\xi)^{1-2\alpha+2\nu}}\,d\xi$$

$$\times \int_c^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr \quad \text{... (3.10)}$$

where,

$$\Phi_1(\xi) = \int_\xi^b \frac{r^{-2\alpha-2\sigma-1}\phi_1(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr \quad \text{... (3.11)}$$

Equation (3.10) is an Abel integral equation and its solution is

$$E(\xi)\,\Phi_1(\xi) = F(\xi) - \frac{\sin(1-2\alpha+2\nu)\,\pi}{\pi}\int_0^a E(y)\,dy$$

$$\times \frac{d}{d\xi}\int_a^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}\,(x-y)^{1-2\alpha+2\nu}}$$

$$\times \int_a^b \frac{r^{-2\alpha-2\sigma-1}\,\phi_1(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr - \frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\int_0^a E(y)\,dy$$

$$\frac{d}{d\xi}\int_a^\xi \times \frac{dx}{(\xi-x)^{2\alpha-2\nu}\,(x-y)^{1-2\alpha+2\nu}}\int_c^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr$$

$$-\ \frac{\sin(1-2\alpha+2\nu)\,\pi}{\pi}\,\frac{d}{d\xi}\int_a^\xi E(y)\,dy\int_y^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}\,(x-y)^{1-2\alpha+2\nu}}$$

$$\times \int_c^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr \quad \text{... (3.12)}$$

where,

$$F(\xi) = \frac{\sin(1-2\alpha+2\nu)\,\pi}{\pi}\,\frac{\Gamma(2\alpha-2\nu)\,\Gamma(2\beta-2\nu)}{2^{2\alpha-2\nu}}\,\frac{d}{d\xi}\int_a^\xi \frac{e^x M(x)}{(\xi-x)^{2\alpha-2\nu}}\,dx \quad \text{... (3.13)}$$

with the help of equation (3.11) we obtain

$$\int_a^b \frac{r^{-2\alpha-2\sigma-1}\,\phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\,dr = \frac{\sin(1-2\beta+2\nu)\pi}{\pi(a-y)^{-2\beta+2\nu}}\int_a^b \frac{\Phi_1(\xi)}{(\xi-a)^{2\beta-2\nu}(\xi-y)}\,d\xi \text{.. (3.14)}$$

Now let us consider

$$\Phi_2(\xi) = \int_\xi^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-\xi)^{1-2\beta+2\nu}}\,dr. \quad \text{... (3.15)}$$

Therefore,

$$\int_c^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-y)^{1-2\beta+2\nu}}\,dr = \frac{\sin(1-2\beta+2\nu)\,\pi}{\pi\,(c-y)^{-2\beta+2\nu}}\int_c^d \frac{\Phi_2(\xi)}{(\xi-c)^{2\beta-2\nu}\,(\xi-y)}\,d\xi \qquad \dots (3.16)$$

Making an appeal to the above results (3.14), (3.15) and (3.16), we derive from (3.12)

$$E(\xi)\,\Phi_1(\xi) = F(\xi) - \int_a^b P(t,\xi)\,\Phi_1(t)\,dt - \int_c^d Q(t,\xi)\,\Phi_2(t)\,dt -$$

$$\frac{\sin(1-2\beta+2\nu)\,\pi}{\pi}\,\frac{d}{d\xi}\int_a^\xi \frac{E(y)}{(c-y)^{-2\beta+2\nu}}\,dy\int_c^d \frac{\Phi_2(t)}{(t-c)^{2\beta-2\nu}\,(t-y)}\,dt \qquad \dots (3.17)$$

where,

$$\frac{d}{d\xi}\int_a^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}\,(x-y)^{1-2\alpha+2\nu}} = \frac{(a-y)^{2\alpha-2\nu}}{(\xi-y)\,(\xi-a)^{2\alpha-2\nu}}\,, \qquad \dots (3.18)$$

$$\int_t^\xi \frac{dx}{(\xi-x)^{2\alpha-2\nu}\,(x-t)^{1-2\alpha+2\nu}} = \frac{\pi}{\sin(1-2\alpha+2\nu)\,\pi}\,, \qquad \dots (3.19)$$

$$P(t,\xi) = \frac{\sin(1-2\alpha+2\nu)\,\pi\,.\,\sin(1-2\beta+2\nu)\,\pi}{\pi^2\,(\xi-a)^{2\alpha-2\nu}}\,.\,\frac{1}{(t-a)^{2\beta-2\nu}}$$

$$\int_0^a \frac{(a-y)^{2\alpha+2\beta-4\nu}\;E(y)}{(\xi-y)\,(t-y)}\,dy \qquad \dots (3.20)$$

and

$$Q(t,\xi) = \frac{\sin(1-2\alpha+2\nu)\,\pi\,.\,\sin(1-2\beta+2\nu)\,\pi}{\pi^2\,(\xi-a)^{2\alpha-2\nu}}\,.\,\frac{1}{(t-c)^{2\beta-2\nu}}$$

$$\times\int_0^a \frac{(a-y)^{2\alpha-2\nu}\;(c-y)^{2\beta-2\nu}\;E(y)}{(\xi-y)\,(t-y)}\,dy \qquad \dots (3.21)$$

Now starting with equation (3.4), we derive

$$E(\xi)\,\Phi_2(\xi) = G(\xi) - \frac{\sin(1-2\alpha+2\nu)\,\pi}{\pi}\int_0^c E(y)\,dy$$

$$\times\,\frac{d}{dy}\int_c^\xi \frac{dx}{(x-y)^{1-2\alpha+2\nu}\,(\xi-x)^{2\alpha-2\nu}}$$

$$\times\int_c^d \frac{r^{-2\alpha-2\sigma-1}\,\phi_2(r)}{(r-y)^{1-2\beta+2\nu}}\,dr - \frac{\sin(1-2\alpha+2\nu)\,\pi}{\pi}\int_0^a E(y)\,dy\,\frac{d}{d\xi}\int_c^\xi$$

$$\times\quad \frac{dx}{(x-y)^{1-2\alpha+2\nu}\,(\xi-x)^{2\alpha-2\nu}}\int_a^b \frac{r^{-2\alpha-2\sigma-1}\,\phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\,dr$$

$$- \frac{\sin(1-2\alpha+2\nu)\pi}{\pi} \int_a^b E(y)\,dy\ \frac{d}{d\xi} \int_c^\xi \frac{dx}{(x-y)^{1-2\alpha+2\nu}\,(\xi-x)^{2\alpha-2}}$$

$$\times \int_y^b \frac{r^{-2\alpha-2\sigma-1}\,\phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\,dr \qquad \ldots (3.22)$$

where

$$G(\xi) = \frac{\sin(1-2\alpha+2\nu)\pi}{\pi}\,\frac{\Gamma(2\alpha-2\nu)\,\Gamma(2\beta-2\nu)}{2^{2\alpha-2\nu}}\,\frac{d}{d\xi}\int_c^\xi \frac{e^x N(x)}{(\xi-x)^{2\alpha-2\nu}}\,dx$$

$$\ldots (3.23)$$

With the help of results (3.15), (3.16), (3.18) and (3.19), we also obtain

$$E(\xi)\,\Phi_2(\xi) = G(\xi) - \int_c^d R(t,\xi)\,\Phi_2(t)\,dt - \int_a^b S(t,\xi)\,\Phi_1(t)\,dt$$

$$- \frac{\sin(1-2\alpha+2\nu)\pi}{\pi(\xi-c)^{2\alpha-2\nu}} \int_a^b \frac{(c-y)^{2\alpha-2\nu}}{(\xi-y)}\,E(y)\,dy \int_y^b \frac{r^{-2\alpha-2\sigma-1}\,\phi_1(r)}{(r-y)^{1-2\beta+2\nu}}\,dr$$

$$\ldots (3.24)$$

where,

$$R(t,\xi) = \frac{\sin(1-2\alpha+2\nu)\pi\,.\,\sin(1-2\beta+2\nu)\pi}{\pi^2(\xi-c)^{2\alpha-2\nu}(t-c)^{2\beta-2\nu}} \int_0^c \frac{(c-y)^{2\alpha+2\beta-4\nu}\,E(y)}{(\xi-y)\,(t-y)}\,dy$$

$$\ldots (3.25)$$

$$S(t,\xi) = \frac{\sin(1-2\alpha+2\nu)\pi\,.\,\sin(1-2\beta+2\nu)\pi}{\pi^2(\xi-c)^{2\alpha-2\nu}\,(t-a)^{2\beta-2\nu}}$$

$$\int_0^a \frac{(c-y)^{2\alpha-2\nu}\,(a-y)^{2\beta-2\nu}\,E(y)}{(\xi-y)\,(t-y)}\,dy \qquad \ldots (3.26)$$

The equations (3.17) and (3.24) are simultaneous Fredholm integral equations of second kind, which determine $\Phi_1(\xi)$ and $\Phi_2(\xi)$. $\phi_1(r)$ and $\phi_2(r)$ can then be found from equations (3.11) and (3.15) respectively and hence the coefficient A_n can be determined from the equation (3.2).

4. PARTICULAR CASES

From the above results, it is easy to derive the solutions of the corresponding dual, triple and quadruple series equations involving generalised Bateman K-function.

REFERENCES

[1] Cooke, J.C., Triple integral equations, *Quart J. Mech. Appl. Math.* **16** (1963), 193-203.

[2] Dwivedi, A.P. and Trivedi, T.N., Triple series equations involving generalised

Bateman *K*-functions, *Indian J. Pure Appl. Math.*, **7** (1976) 320-327

[3] Dwivedi, A.P. and Trivedi, T.N., Quadruple series equations involving generalised Bateman *K*-functions, *Labdev.J. of. Sci. and Tech.*, **13-A** (1975), 168-174.

[4] Dwivedi, A.P. and Pandey, S., Certain five series equations involving generalised Bateman *K*- functions, *Acta. Ciencia. Indica* ,**17** M. 2 (1991), 301-308.

[5] Lowndes, J.S. Some dual series equations involving Laguerre polynomials., *Pac. J. Math.* **25,** 1 (1968) 123-127

[6] Lowndes, J.S. Triple series equations involving Laguerre- polynomial, *Pac. J. Math.* **29**, No. 1 (1969), 167-173.

[7] Noble, B. Some dual series equations involving Jacobi polynomials, *Proc. Camb. Phil. Soc.* **59,** (1963) 363-371.

[8] Sneddon, I.N., *Mixed Boundary Value Problems in Potential Theory*, North-Holland Pub.Company. Amsterdam (1966).

[9] Srivastava, H.M., A pair of dual series equations involving generalised Bateman K- function, *Nederl. Akad. Wetensch. Proc. Ser. A*, **75** *Indag. Math* **34** (1972), 53-61.

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CERTAIN FIVE SERIES EQUATIONS INVOLVING GENERALISED LAGUERRE POLYNOMIALS

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ABSTRACT

Solution of certain five series equations, involving generalised Laguerre polynomials, has been obtained in this paper.

1. INTRODUCTION

Dual and triple series equations arise in many mixed boundary value problems of mathematical physics [8]. The above equations are the generalisation of dual, triple and quadruple series equations, considered earlier by [9], [2] and [3]. The object of this paper is to obtain the solution of five series equations involving generalised Laguerre polynomials. Recently various authors [1], [4], [5], [6] and [7] have considered other series equations involving different special functions.

2. THE EQUATIONS

We shall solve the following system of five series equations involving generalised Laguerre polynomials.

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)} L_n^{\nu}(x) = \begin{cases} f_1(x) \ , \ 0<x<a \\ f_3(x) \ , \ b<x<c \\ f_5(x) \ , \ d<x<\infty \end{cases} \qquad \text{... (2.1)}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{\sigma}(x) = \begin{cases} f_2(x) \ , \ a<x<b \\ f_4(x) \ , \ c<x<d \end{cases} \qquad \text{... (2.2)}$$

where $L_n^{\alpha}(x)$ is the generalised Laguerre polynomial, A_n is unknown coefficient, $f_1(x), f_2(x)$, $f_3(x)$, $f_4(x)$ and $f_5(x)$ are known functions and the parameters $\alpha, \beta, \nu, \sigma$ all are > -1. It is shown here that the problem of solving five series equations can be reduced to that of solving simultaneous Fredholm integral equations of second kind.

3. SOLUTION OF FIVE SERIES EQUATIONS

Let us suppose

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)} L_n^{\nu}(x) = \begin{cases} \phi_1(x) \; , \; a < x < b \\ \phi_2(x) \; , \; c < x < d \end{cases} \qquad \text{... (3.1)}$$

Using orthogonality relation, we get from (2.1) and (3.1)

$$A_n = \frac{\Gamma(n+1)\,\Gamma(\alpha+n+1)}{\Gamma(\nu+n+1)} \left[\int_0^a f_1(x) + \int_a^b \phi_1(x) + \int_b^c f_3(x) + \int_c^d \phi_2(x) \right.$$

$$\left. + \int_d^{\infty} f_5(x) \quad x^{\nu} e^{-x} L_n^{\nu}(x)\, dx \right. \qquad \text{... (3.2)}$$

Substituting this expression for A_n from (3.2) in (2.2) we have

$$\int_a^b e^{-r}\, S(r,x)\, \phi_1(r)\, dr + \int_c^d e^{-r}\, S(r,x)\, \phi_2(r)\, dr = \begin{cases} M(x) \; , \; a < x < b & \text{... (3.3)} \\ N(x) \; , \; c < x < d & \text{... (3.4)} \end{cases}$$

where

$$M(x) = x^{\sigma} f_2(x) - \int_0^a e^{-r}\, S(r,x)\, f_1(r) dr - \int_b^c e^{-r}\, S(r,x)\, f_3(r)\, dr$$

$$- \int_d^{\infty} e^{-r}\, S(r,x)\, f_5(r)\, dr \qquad \text{... (3.5)}$$

$$N(x) = x^{\sigma} f_4(x) - \int_0^a e^{-r}\, S(r,x)\, f_1(r)\, dr - \int_b^c e^{-r}\, S(r,x)\, f_3(r)\, dr$$

$$- \int_d^{\infty} e^{-r}\, S(r,x)\, f_5(r)\, dr \qquad \text{... (3.6)}$$

and

$$\Gamma(\lambda)\,\Gamma(1-\lambda)\, S(r,x) = \Gamma(\lambda)\,\Gamma(1-\lambda)\, r^{\sigma} x^{\nu}$$

$$\sum_{n=0}^{\infty} \times \frac{\Gamma(\beta+n+1)\,\Gamma(n+1)}{\Gamma(\alpha+n+1)\,\Gamma(\sigma+n+1)} L_n^{\sigma}(r)\, L_n^{\nu}(x) \qquad \text{... (3.7)}$$

$$= a^{*} \int_0^t \eta(\xi)\, (r-\xi)^{\lambda-1}\, (x-\xi)^{\lambda+\nu-\sigma-1}\, d\xi \qquad \text{... (3.8)}$$

$$= a^{*}\; S_r(r,x) \qquad \text{... (3.9)}$$

It is further assumed that a^{*} is independent of n.

Starting with equation (3.3) and making some calculations, we derive

$$\int_a^x \frac{\eta(y)\, \Phi_1(y)}{(x-y)^{1+\nu-\lambda-\sigma}}\, dy = \frac{\Gamma(\lambda)\,\Gamma(1-\lambda)}{a^{*}} M(x) - \int_0^a \frac{\eta(y)}{(x-y)^{1+\nu-\lambda-\sigma}}\, dy$$

$$\times \int_a^b \frac{e^{-r}\,\phi_1(r)}{(x-y)^{1-\lambda}}\,dr - \int_0^x \frac{\eta(y)}{(x-y)^{1+\nu-\lambda-\sigma}}\,dy \int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-y)^{1-\lambda}}\,dr \quad \text{... (3.10)}$$

where

$$\Phi_1(y) = \int_y^b \frac{e^{-r}\,\phi_1(r)}{(r-y)^{1-\lambda}}\,dr, \quad 0 < \lambda < 1 \qquad \text{... (3.11)}$$

Equation (3.10) is an Abel integral equation and its solution is

$$\eta(y)\,\Phi_1(y) = F(y) - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi} \int_0^a \eta(\xi)\,d\xi \times \frac{d}{dy} \int_a^y \frac{dx}{(y-x)^{\lambda+\sigma-\nu}(x-\xi)^{1+\nu-\lambda-\sigma}}$$

$$\times \int_a^b \frac{e^{-r}\,\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi} \int_0^a \eta(\xi)\,d\xi\,\frac{d}{dy} \int_a^y$$

$$\times \frac{dx}{(y-x)^{\lambda+\sigma-\nu}\,(x-\xi)^{1+\nu-\lambda-\sigma}} \int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-\xi)^{1-\lambda}}dr - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\,\frac{d}{dy}$$

$$\times \int_a^y \eta(\xi)\,d\xi \int_\xi^y \frac{dx}{(y-x)^{\lambda+\sigma-\nu}\,(x-\xi)^{1+\nu-\lambda-\sigma}} \int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-\xi)^{1-\lambda}}\,dr. \quad \text{... (3.12)}$$

where

$$F(y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\,\frac{\Gamma(\lambda)\,\Gamma(1-\lambda)}{a^*}\,\frac{d}{dy} \int_a^y \frac{M(x)}{(y-x)^{\lambda+\sigma-\nu}}\,dx.. \quad \text{(3.13)}$$

With the help of equation (3.11), we obtain

$$\int_a^b \frac{e^{-r}\,\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr = \frac{\sin(1-\lambda)\pi}{\pi\,(a-\xi)^{-\lambda}} \int_a^b \frac{\Phi_1(y)}{(y-a)^{\lambda}\,(y-\xi)}\,dy \qquad \text{... (3.14)}$$

Now let us consider

$$\Phi_2(y) = \int_y^d \frac{e^{-r}\,\phi_2(r)}{(r-y)^{1-\lambda}}\,dr\,. \qquad 0 < \lambda < 1 \qquad \text{... (3.15)}$$

Therefore

$$\int_c^d \frac{e^{-r}\,\phi_2(r)}{(r-\xi)^{1-\lambda}}\,dr = \frac{\sin(1-\lambda)\pi}{\pi\,(c-\xi)^{-\lambda}} \int_c^d \frac{\Phi_2(y)}{(y-c)^{\lambda}\,(y-\xi)}\,dy \qquad \text{... (3.16)}$$

Making an appeal to the results (3.14), (3.15) and (3.16), we derive from (3.12)-

$$\eta(y)\,\Phi_1(y) = F(y) - \int_a^b P(t,y)\,\Phi_1(t)\,dt - \int_c^d Q(t,y)\,\Phi_2(t)\,dt - \frac{\sin(1-\lambda)\pi}{\pi}$$

$$\times \frac{d}{dy}\int_a^y \frac{\eta(\xi)}{(c-\xi)^{-\lambda}}\,d\xi \int_c^d \frac{\Phi_2(t)}{(t-c)^{\lambda}(t-\xi)}\,dt \qquad \text{... (3.17)}$$

where

$$\frac{d}{dy}\int_a^y \frac{dx}{(y-x)^{\lambda+\sigma-\nu}(x-\xi)^{1+\nu-\lambda-\sigma}} = \frac{(a-\xi)^{\lambda+\sigma-\nu}}{(y-\xi)(y-a)^{\lambda+\sigma-\nu}} \qquad \text{... (3.18)}$$

$$\int_t^{\xi} \frac{dx}{(\xi-x)^{\lambda-\nu+\sigma}(x-t)^{1-\sigma-\lambda+\nu}} = \frac{\pi}{\sin(1+\nu-\lambda-\sigma)\pi} \qquad \text{... (3.19)}$$

$$P(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi\,.\,\sin(1-\lambda)\pi}{\pi^2(y-a)^{\lambda+\sigma-\nu}}\cdot\frac{1}{(t-a)^{\lambda}}\int_0^a \frac{(a-\xi)^{2\lambda+\sigma-\nu}\,\eta(\xi)}{(y-\xi)(t-\xi)}\,d\xi$$

$$\text{... (3.20)}$$

$$Q(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi\,.\,\sin(1-\lambda)\pi}{\pi^2(y-a)^{\lambda+\sigma-\nu}}\cdot\frac{1}{(t-c)^{\lambda}}$$

$$\int_0^a \frac{(a-\xi)^{\lambda+\sigma-\nu}(c-\xi)^{\lambda}\,\eta(\xi)}{(y-\xi)(t-\xi)}\,d\xi \qquad \text{... (3.21)}$$

Now starting with equation (3.4), we derive

$$\eta(y)\,\Phi_2(y) = G(y) - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\int_0^c \eta(\xi)\,d\xi\,\frac{d}{dy}\int_c^y$$

$$\times \frac{dx}{(x-\xi)^{1+\nu-\lambda-\sigma}(y-x)^{\lambda+\sigma-\nu}}\int_c^d \frac{e^{-r}\phi_2(r)}{(r-\xi)^{1-\lambda}}\,dr - \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\,.$$

$$\times \int_0^a \eta(\xi)\,d\xi\,\frac{d}{dy}\int_c^y \frac{dx}{(x-\xi)^{1+\nu-\lambda-\sigma}(y-x)^{\lambda+\sigma-\nu}}\int_a^b \frac{e^{-r}\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr -$$

$$\frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi}\int_a^b \eta(\xi)\,d\xi\,\frac{d}{dy}\int_c^y \frac{dx}{(x-\xi)^{1+\nu-\lambda-\sigma}(y-x)^{\lambda+\sigma-\nu}}$$

$$\times \int_{\xi}^b \frac{e^{-r}\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr \qquad \text{... (3.22)}$$

where

$$G(y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi} \, \frac{\Gamma(\lambda)\,\Gamma(1-\lambda)}{a^*} \, \frac{d}{dy} \int_c^y \frac{N(x)}{(y-x)^{\lambda+\sigma-\nu}}\, dx \quad \text{.. (3.23)}$$

With the help of results (3.15), (3.16), (3.18) and (3.19), we also obtain

$$\eta(y)\,\Phi_2(y) = G(y) - \int_c^d R(t,y)\,\Phi_2(t)\,dt - \int_a^b S(t,y)\,\Phi_1(t)\,dt -$$

$$\frac{\sin(1+\nu-\lambda-\sigma)\pi}{\pi(y-c)^{\lambda+\sigma-\nu}} \int_a^b \frac{(c-\xi)^{\lambda+\sigma-\nu}\,\eta(\xi)}{(y-\xi)}\,d\xi \int_\xi^b \frac{e^{-r}\,\phi_1(r)}{(r-\xi)^{1-\lambda}}\,dr \quad \text{... (3.24)}$$

where

$$R(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi \,.\, \sin(1-\lambda)\pi}{\pi^2 (y-c)^{\lambda+\sigma-\nu}\,(t-c)^{\lambda}} \int_0^c \frac{(c-\xi)^{2\lambda+\sigma-\nu}\,\eta(\xi)}{(y-\xi)\,(t-\xi)}\,d\xi \quad \text{... (3.25)}$$

$$S(t,y) = \frac{\sin(1+\nu-\lambda-\sigma)\pi \,.\, \sin(1-\lambda)\pi}{\pi^2 (y-c)^{\lambda+\sigma-\nu}\,(t-a)^{\lambda}} \int_0^a \frac{(c-\xi)^{\lambda+\sigma-\nu}\,(a-\xi)^{\lambda}\,\eta(\xi)}{(y-\xi)\,(t-\xi)}\,d\xi \quad \text{... (3.26)}$$

The equations (3.17) and (3.24) are simultaneous Fredholm integral equations of second kind, which determine $\Phi_1(y)$ and $\Phi_2(y)$. $\phi_1(r)$ and $\phi_2(r)$ can then be found from equations (3.11) and (3.15) respectively and hence the coefficients A_n can be determined from the equation (3.2).

4. PARTICULAR CASES

From the above results, it is easy to derive the solutions of the corresponding dual, triple and quadruple series equation involving generalised Laguerre polynomials.

REFERENCES

[1] Cooke, J.C. , Triple integral equations, *Quart.* J. Mech. Appl. Math., 16 (1963), 193-203.

[2] Dwivedi, A. P. and Trivedi, T. N., Some triple series equations involving generalised Laguerre polynomials, Indian. *J. Pure. Appl. Math.*, **5**, No 7 (1974), 674

[3] Dwivedi, A. P. and Trivedi, `T. N., Quadruple series equations involving generalised Laguerre polynomials, *J. Math. Phy. Sci.*, **9**, No. 1 (1975) 83-91

[4] Dwivedi A. P. and Pandey, S., Certain five series equations involving generalised Bateman *K*-function., *Acta Ciencia Indica*, **17**, M, 2 (1991), 301-308

[5] Lowndes, J. S. , Some dual series equations involving Laguerre polynomials, *Pac. J. Math.*, **25**, No. 1 (1968), 123-127.

[6] Lowndes, J. S. , Triple series equations involving Laguerre polynomials, *Pac. J. Math.*, **29**, No. 1 (1969), 167-173.

[7] Noble, B, Some dual series equations involving Jacobi polynomials, *Proc. Camb. Phil. Soc.*, **59** (1963), 363-371

[8] Sneddon, I.N., *Mixed Boundary Value Problems in Potential Theory*.. North-Holland Publ. Company, Amsterdam, (1966).

[9] Srivastava, H. M. Dual series relations involving generalised Laguerre polynomials, *J. Math. Anal. Appl.*, **31** (1970), 587-594

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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

(1, 1 RINGS)

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ABSTRACT

A simple not associative (1, 1) ring, with commutators in the left nucleus, has no proper left ideals.

A left primitive (1, 1) ring, with commutators in the left nucleus and associators in the centre, is either associative or simple with a right identity element. Let A be a left ideal of (1, 1) ring R with commutators in the nucleus N. If A is maximal and nil, then A is a two-sided ideal. If A *is* minimal, then it is either a two-sided ideal, or the ideal it generates is contained in N.

1. INTRODUCTION

A (1, 1) ring R is a non-associative ring in which the following identities hold:

(1) $(x, y, z) = (x, z, y)$,

(2) $(x, y, z) + (y, z, x) + (z, x, y) = 0$,

for all x, y, z in R where the asociator $(x, y, z) = (x\,y)\,z - x\,(y\,z)$.

(R, R, R) is linear span of all associators in R .Kleinfeld [3] proved that a simple (1, 1) ring with a non-zero idempotent is associative.

We shall denote the left nucleus by L, nucleus by N and centre by U. Thus L consists of all elements l in R such that $(l, R, R) = (0)$. In (1, 1) ring R, N consists of all elements n in R such that $(R, n, R) = (0)$. By virtue of (1), $(R, R, n) = (0)$. Using (2), we have $(n, R, R) = (0)$. So $N \subseteq L$. It is well known [4] that N is an associative subring of R. U consists of all elements u in R such that $(u, R) = (0)$, where the commutator $(x, y) = xy - yx$. (R, R) is linear span of all commutators in R.

2. SIMPLE RINGS

Lemma 1. If R is simple, not associative and $(R, R) \subseteq L$, then R has no proper left ideals.

Proof. Suppose that A is a nonzero left ideal of R. Then $(R, R, A) \subseteq A$; hence from (1), we have $(R, A, R) \subseteq A$ and from (2), we have $(A, R, R) \subseteq A$. We shall show that $A = R$. $A + AR$ is a two sided ideal of R, since

$$(A + AR)\,R \subseteq AR + (AR)\,R \subseteq AR + A\,(RR) + (A, R, R) \subseteq AR + A,$$

and

$$R\,(A + AR) \subseteq RA + R\,(AR) \subseteq A + (RA)\,R + (R, A, R) \subseteq A + AR.$$

Since $A \neq (0)$ and R is simple, $R = A + AR$. Hence

$$\begin{aligned}(R, R, R) &\subseteq (A + AR, R, R) \subseteq (A, R, R) + (AR, R, R)\\ &\subseteq (A, R, R) + (RA, R, R) \subseteq (A, R, R) + (A, R, R)\\ &\subseteq (A, R, R) \subseteq A.\end{aligned}$$

Since $RA \subseteq A$, $(R, R, R) + R(R, R, R) \subseteq A$. But it is known [2] that $(R, R, R) + R\,(R, R, R)$ is a two-sided ideal of R, hence it is equal to R since R is not associative. Therefore $A = R$.

In any nonassociative ring we have

$$(xy, z) - x\,(y, z) - (x, z)\,y = (x, y, z) - (x, z, y) + (z, x, y).$$

Using (1), the above identity becomes

$$(3) \qquad (x\,y, z) - x\,(y, z) - (x, z)\,y = (z, x, y).$$

Lemma 2. Suppose R is simple, not associative and $(R, R) \subseteq N$. If T is a subset of N, then $(T, R) = (0)$.

Proof. First we show that $(T, R) + R(T, R)$ is a left ideal of R.

$$\begin{aligned}R\,((T, R) + R\,(T, R)) &= R\,(T, R) + R\,(R\,(T, R))\\ &\subseteq R\,(T, R) + (R, R)\,(T, R) + (R, R\,(T, R))\\ &\subseteq R\,(T, R) + R\,(T, R) \;(\because (T, R) \subseteq (R, R) \subseteq N)\\ &\subseteq R\,(T, R) \subseteq (T, R) + R\,(T, R).\end{aligned}$$

Therefore $(T, R) + R\,(T, R)$ is a left ideal of R.

By Lemma 1, either $(T, R) + R\,(T, R) = R$ or $(T, R) + R\,(T, R) = (0)$.

Suppose $(T, R) + R\,(T, R) = R$. Let $y \in T$ and $x, z, \in R$. Then by (3)

$$(x, z)\,y = (xy, z) - x\,(y, z) - (z, x, y).$$

$y \in T \subseteq N$ implies that $(z, x, y) = 0$. Therefore,

$$(x, z)\,y = (x\,y, z) - x\,(y, z)$$

or

$$x\,(y, z) = (x\,y, z) - (x, z)\,y.$$

Now $y \in T \subseteq N$, $(x, z) \in (R, R) \subseteq N$ and N is a subring of R. This implies that $(x, z)\,y \in N$. Also $(xy, z) \in (R, R) \subseteq N$. Thererfore, $x\,(y, z) \in N$. This implies that $R\,(T, R) \subseteq N$. Also $(T, R) \subseteq (R, R) \subseteq N$. So $(T, R) + R\,(T, R) \subseteq N$. That is, $R \subseteq N$. Hence R is associative. This contradicts the hypothesis. So $(T, R) + R\,(T, R) = (0)$ which implies that $(T, R) = (0)$.

Theorem 3. Suppose R is simple, not associative and $(R, R) \subseteq N$. Then $(R, R) \subseteq U$.

Proof. Taking $T = (R, R)$ in lemma 2, we see that $T \subseteq N$.

Therefore $(T, R) = (0)$. This implies that $((R, R), R) = (0)$. Hence

$$(R, R) \subseteq U.$$

3. CONSTRUCTION OF IDEALS

Lemma 4. Let A be a left ideal of R. Then

(i) $S \left\{ s \in A : sR \subseteq A \right\}$ is a two-sided ideal of R.

(ii) If $(R, R, R) \subseteq U$, then $(R, A, R) \subseteq S$.

Proof. (i) For any $s \in S$, $x \in R$, consider sx and xs. Let $y \in R$. Then

$$\begin{aligned} (s\,x)\,y &= (s, x, y) + s\,(x\,y) \\ &= -\ (x, y, s) -\ (y, s, x) + s\,(x\,y) && \text{(using (2))} \\ &= -\ (x, y, s) -\ (y, x, s) + s\,(x\,y) && \text{(using (1)).} \end{aligned}$$

A is a left ideal and $s \in S$ implies that

$$-\ (x, y, s) -\ (y, x, s) + s\,(x\,y) \in A.$$

Hence $(sx)\,y \in A$. Also $sR \subseteq A$ implies that $sx \in A$. Hence $sx \in$ [illegible] Now,

$$\begin{aligned} (x\,s)\,y &= (x, s, y) + x\,(s\,y) \\ &= (x, y, s) + x\,(s\,y) && \text{(using (1)).} \end{aligned}$$

Again using the fact that A is a left ideal and $s \in S$ in the above equation, we get $(x\,s)y \in A$. Hence $xs \in S$. So S is a two sided ideal of R.

(ii) Let $x, w, z \in R$ and $a \in A$. Then

$$\begin{aligned} (w, a, x)\,z &= (w, x, a)\,z && \text{(using (1)).} \\ &= z\,(w, x, a). \end{aligned}$$

Again, since A is a left ideal, $z(w, x, a) \in A$. Therefore $(w, a, x)\,z \in A$. Hence $(R, A, R) \subseteq S$.

Theorem 5. Suppose $(R, R) \subseteq L$ and $(R, R, R) \subseteq U$. If R has a maximal left ideal $A \neq (0)$, which contains no two-sided ideal of R other than (0), then R is associative.

Proof. By Lemma 4, S is a two-sided ideal of R contained in A. Therefore $S = (0)$ and hence $(R, A, R) = (0)$. Using (1), we get $(R, R, A) = (0)$. Using (2), we obtain $(A, R, R) = (0)$, Now as in the proof of Lemma 1, $A + AR$ is a two-sided ideal of R. Since $A \subset A + AR$, we must have $A + AR = R$, Hence as in the proof of Lemma 1, $(R, R, R) \subseteq (A, R, R) = (0)$. That is, R is associative.

Theorem 6. Suppose $(R, R) \subseteq N$ and let A be a left ideal of R.

(a) If A is maximal and nil, then A is a two-sided ideal of R.

(b) If A is maximal, then either A is a two-sided ideal of R or the ideal generated in R by A is contained in N.

Proof. The motivation for this theorem and its method of proof are same as Theorem 1 in [1]. In fact, if one exchanges (1, 1) for right alternative, A for L, R for A, and the identities employed, then the notaiton and write up of the proof in [1] is virtually identical to the one here. However, in proof of part (b), one may argue as follows:

Clearly $(R, R, A) \subseteq A$.

A straight forward verification shows that any ring satisfies

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w\,(x, y, z) + (w, x, y)\,z$$

which is known as the Teichmüller identity. Put $w = x, x = y$, $y = a, z, = r$ in this identity where $a \in A$, $r \in R$. We get

$$(x, y, a)\,r = (x\,y, a, r) - (x, ya, r) + (x, y, ar) - x\,(y, a, r)$$

$$= (xy, r, a) - (x, r, ya) + (x, y, ra) - x\,(y, r, a).$$

So $(x, y, a)r \in A$. This implies that $(R, R, A) \subseteq A' = 0$, i.e. $A \subseteq N$.

4. LEFT PRIMITIVE RINGS

Definition. A left ideal A of R is called regular if there exists an element $g \in R$ such that $x - xg \in A$ for all $x \in R$. R is called **left pimitive** if it contains a regular maximal ideal which contains no two-sided ideal of R other than (0).

Theorem 7. Suppose $(R, R) \subseteq L$ and $(R, R, R) \subseteq U$. If R is a left primitive ring, then either R is associative or it is simple with a right identity element.

Proof. Let A be a regular maximal left ideal which contains no two-sided ideals of R other than (0). If $A \neq (0)$, then by Theorem 5, R is associative. Thus assume that (0) is a maximal regular left ideal of R, which contains no two-sided ideal of R other than (0). (0) is maximal implies that R has no proper ideal. Therefore R is simple. Since (0) is regular, there exists $g \in R$ such that $x - xg \in (0)$ for all $x \in R$. That is, $x = xg$ for all $x \in R$ and g is a right identity element. Hence R is simple with a right identity element.

REFERENCES

[1] I.R. Hentzel and H.F. Smith, Semiprime locally $(-1, 1)$ rings with minimal condition, *Algebras Groups Geom.* **2** (1985), 26-52.

[2] E. Kleinfeld, Assosymmetric rings, *Proc. Amer. Math. Soc.* **88** (1957), 983-986.

[3] E. Kleinfeld, Simple algebras of type (1, 1) are associative, *Canad. J. Math.* **13** (1961), 129-148.

[4] R.D. Schafer, An introduction to nonassociative algebras, *Pure and Appl. Math.*, **22**, Academic Press, New York and London, 1966.

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MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST AND ITS APPLICATIONS IN TWO BOUNDARY VALUE PROBLEMS

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ABSTRACT

In the present paper first we evaluate an interesting integral involving multiple hypergeometric function of Srivastava and Daoust [5,6,7] and then we make its applications to solve two boundary value problems on (i) heat conduction in a rod (ii) deflection of vibrating string, under certain conditions and to establish an expansion formula involving above multiple hypergeometric function.

1. INTRODUCTION. Recently Chandel and Yadava [1] and Chandel and Gupta [2], have employed the hypergometric function of Srivastava and Daoust [5, 6, 7] (also see for modified form Srivastava and Karlsson [8, p.37 eqs. (21) to (23)]), in different problems on heat conduction. In the present paper, first we evaluate a new integral involving the multiple hypergeometric function of Srivastava and Daoust [5, 6, 7] defined by

$$(1.1)\ S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\dots,x_n\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{\prod\limits_{j=1}^{A}\Gamma\left(a_j+\sum\limits_{i=1}^{n} m_i\,\theta_j^{(i)}\right)\prod\limits_{j=1}^{B'}\Gamma(b'_j+m_1\phi'_j)\dots\prod\limits_{j=1}^{B^{(n)}}\Gamma(b_j^{(n)}+m_n\phi_j^{(n)})}{\prod\limits_{j=1}^{C}\Gamma\left(c_j+\sum\limits_{i=1}^{n} m_i\,\psi_j^{(i)}\right)\prod\limits_{j=1}^{D'}\Gamma(d'_j+m_1\delta'_j)\dots\prod\limits_{j=1}^{D^{(n)}}(d_j^{(n)}+m_n\,\delta_j^{(n)})}$$

$$\frac{m_1 x_1}{m_1\,!}\ \cdots\ \frac{m_n x_n}{m_n\,!}$$

where $\theta_j^{(i)}, j=1,\dots,A$; $\phi_j^{(i)}, j=1,\dots,B^{(i)}$; $\psi_j^{(i)}, j=1,\dots,C$;

$\delta_j^{(i)}, j=1,\dots,D^{(i)}$; $1\le i\le n$, are real and positive and (a) is taken to abbreviate the sequence of A parameters $a\ a_1, \ldots, a_A$, $b^{(i)}$ abbreviates the sequence of $B^{(i)}$ parameters $b_1^{(i)}, \ldots, b_{B(i)}^{(i)}$, $i = 1, \ldots, n$ with similar interpretation for (c) and $(d^{(i)})$, $i = 1, \ldots, n$; etc. and its applicastions will be made to derive solution of (i) A problem on heat conduction in a rod (ii) A problem on deflection of vibrating string under certain boundary conditions.

2. FORMULA REOUIRED. In this paper, we shall make aplication of the following modified form of the integral [3, p. 372, (1)]:

$$(2.1)\quad \int_0^L (\sin \frac{\pi x}{L})^{w-1} \sin \frac{\pi x \lambda_m}{L}\, dx$$

$$= \frac{\omega L \sin \pi \lambda_m / 2}{2^{w-1} \Gamma(\frac{w+\lambda_m+1}{2}) \Gamma(\frac{w-\lambda_m+1}{2})}, R_e(w) > 0 .$$

3. INTEGRAL. Making an appeal to (2.1). we obtain

$$(3.1)\quad \int_0^L (\sin \frac{\pi x}{L})^{w-1} \sin \frac{\lambda_m \pi x}{L} S^{(A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left(\begin{matrix} [(a):\theta', \ldots, \theta^{(n)}] : [(b'):\theta'] ;\ldots; [(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi', \ldots, \psi^{(n)}] : [(d'):\delta'] ;\ldots; [(d^{(n)}):\delta^{(n)}]; \end{matrix} \right.$$

$$\left. Z_1 (\sin \frac{\pi x}{L})^{2\xi_1}, \ldots, Z_n (\sin \frac{\pi x}{L})^{2\xi_n} \right) dx$$

$$= \frac{L \sin \frac{\pi \lambda_m}{2}}{2^{\omega-1}} S^{(A+1:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}} \begin{matrix} [(a):\theta',\ldots,\theta^{(n)}], \\ [(c):\psi',\ldots,\psi^{(n)}, \end{matrix}$$

$$[w : 2\xi_1, \ldots, 2\xi_n] :$$

$$\left[\frac{w+1+\lambda_m}{2} : \xi_1, \ldots, \xi_n\right], \left[\frac{w+1-\lambda_m}{2} : \xi_1, \ldots, \xi_n\right] :$$

$$\begin{matrix} [(b'):\phi'] ;\ldots; [(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta'] ;\ldots; [(d^{(n)}):\delta^{(n)}]; \end{matrix} Z_1/4\ \xi_1, \ldots, Z_n/4\xi_n),$$

provided that Re $(w) > 0$,

$$1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0,$$

and all ξ_i are real positive integers. This integral will be used in our further investigations.

PROBLEM 1

4. APPLICATION TO HEAT CONDUCTION IN A ROD. In this section, we consider a problem on outer heat conduction in a rod under certain boundary conditions. If the thermal coefficients are constant and there is no source of thermal energy, then the temperature $u(x,t)$ in a one dimensional rod $0 \leq x \leq L$ satisfies the following heat equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0. \tag{4.1}$$

If we take the following boundary conditions

$$u(0, t) = 0, \tag{4.2}$$

$$\frac{\partial u(L, t)}{\partial x} + hu(L, t) = 0, \tag{4.3}$$

$$u(x, t) \text{ is finite as } t \to \infty, \tag{4.4}$$

and initial condition

$$u(x, 0) = f(x), \tag{4.5}$$

then the solution of partial differential equation (4.1) is given by [4,p.77, (4)]

$$u(x, t) = \sum_{m=1}^{\infty} B_m \sin \frac{\lambda_m \pi x}{L} \exp\left\{-\left(\frac{\pi\lambda_m}{L}\right)^2 kt\right\}, \tag{4.6}$$

where $\lambda_1, \ldots, \lambda_m$ are the roots of the transcendental equation

$$\tan \pi \lambda_m = \frac{\pi\lambda_m}{kL}. \tag{4.7}$$

Now we shall consider the problem of determining u(x, t), where

$$\begin{aligned} u(x, 0) &= f(x) \\ &= \left(\sin \frac{\pi x}{L}\right)^{w-1} S^{(A:B';\ldots;B^{(n)})}_{C:D';\ldots;D^{(n)}} \left(\begin{matrix} [(a):\theta', \ldots, \theta^{(n)}]: \\ [(c):\psi', \ldots, \psi^{(n)}]: \end{matrix} \right. \\ &\quad \left. \begin{matrix} [(b'):\phi']; \ldots; [(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta']; \ldots; [(d^{(n)}):\delta^{(n)}]; \end{matrix} \; Z_1 \left(\sin \frac{\pi x}{L}\right)^{2\xi_1}, \ldots, Z_n \left(\sin \frac{\pi x}{L}\right)^{2\xi_n} \right). \end{aligned} \tag{4.8}$$

5. SOLUTION OF THE PROBLEM. Combining (4.6) and (4.8) and making the use of integral (3.1), we derive

$$B_m = \frac{\pi\lambda_m \sin \frac{\pi \lambda_m}{2}}{2^{w-3}\left[2\pi\lambda_m - \sin 2\pi\lambda_m\right]} \tag{5.1}$$

$$S^{A+1:B';\dots;B^{(n)}}_{C+2:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\ [w:2\xi_1,\dots,2\xi_n]\cdot\\ [(c):\psi',\dots,\psi^{(n)}],\left[\dfrac{w+1+\lambda_m}{2}:\xi_1,\dots,\xi_n\right],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(n)}:\phi^{(n)}];\\ \left[\dfrac{w+1-\lambda_m}{2}:\xi_1,\dots,\xi_n\right]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ Z_1/4\xi_1,\dots,Z_n/4\xi_n\right).$$

where all conditions of (3.1) are satisfied.

Putting the value of B_m from (5.1) in (4.6), we get the following required solution of the problem:

$$(5.2)\quad u(x,t)=\frac{\Pi}{2^{w-3}}\sum_{m=1}^{\infty}\sin\frac{\lambda_m\pi x}{L}\exp\left\{-\left(\frac{\pi\lambda_m}{2}\right)^2 kt\right\}$$

$$\frac{\lambda_m\sin\dfrac{\pi\lambda_m}{2}}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}S^{A+1:B';\dots;B^{(n)}}_{C+2:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\psi',\dots,\psi^{(n)}],\end{matrix}\right.$$

$$[w:2\xi_1,\dots,2\xi_n]:$$

$$\left[\frac{w+1+\lambda_m}{2}:\xi_1,\dots,\xi_n\right],\left[\frac{w+1-\lambda_m}{2}:\xi_1,\dots,\xi_n\right]:$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ Z_1/4\xi_1,\dots,Z_n/4\xi_n\right),$$

where all the conditions of (3.1) hold true.

6. EXPANSION FORMULA. Making an use of (4.8) and (5.1) in (4.6) we derive the following expansion formula

$$(6.1)\quad\left(\sin\frac{\pi x}{L}\right)^{w-1}S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:\\ [(c):\psi',\dots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ Z_1\left(\sin\frac{\pi x}{L}\right)^{2\xi_1},\dots,Z_n\left(\sin\frac{\pi x}{L}\right)^{2\xi_n}\right)$$

$$=\frac{\pi}{2^{w-3}}\sum_{m=1}^{\infty}\frac{\lambda_m\sin\dfrac{\pi\lambda_m}{2}\sin\dfrac{\lambda_m\pi x}{L}}{(2\pi\lambda_m-\sin 2\pi\lambda_m)}S^{(A+1:B';\dots;B^{(n)}}_{C+2:D';\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\psi',\dots,\psi^{(n)}],\end{matrix}\right.$$

$$[w:2\xi_1,\dots,2\xi_n]:$$

$$\left[\frac{w+1+\lambda_m}{2}:\xi_1,\dots,\xi_n\right],\left[\frac{w+1-\lambda_m}{2}:\xi_1,\dots,\xi_n\right]$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b)^{(n)}:\phi^{(n)}];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)};\end{matrix}\ Z_1/4\xi_1,\dots,Z_n/4\xi_n\right).$$

provided that all the conditions of (3.1) are satisfied.

PROBLEM - 2

7. APPLICATION TO HOMOGENEOUS WAVE PROBLEM.

In this section, we shall determine the shape (deflection) $u(x, t)$ of vibrating string. It the deflection due to the weight of string is negligible (usually the case), then $u(x, t)$ satisfies the partial differential equation

$$(7.1) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c} \frac{\partial^2 u}{\partial t^2} \; 0 < x < L, t > 0.$$

Now we assume the boundary conditions

$$(7.2) \qquad u(0, t) = 0, u(L, t) = 0, t > 0,$$

and initial conditions

$$(7.3) \qquad \frac{\partial u}{\partial t}(x, 0) = g(x) \text{ (initial velocity)}$$

and

$$(7.4) \qquad u(x, 0) = f(x) .$$

Then the solution of partial differential equation (7.1) is given by

$$(7.5) \qquad u(x, t) = \sum_{m=1}^{\infty} \left(a_m \cos \frac{\pi \lambda_m ct}{L} + b_m \sin \frac{\pi \lambda_m ct}{L} \sin \frac{n \lambda_m x}{L}, \right)$$

Now we consider the problem of determining $u(x,t)$, where $u(x, 0)$ $(=f(x))$ is given by (4.8) while

$$(7.6) \qquad g(x) = \left(\sin \frac{\pi x}{L} \right)^{w'-1} S^{E ; F' ; \dots ; F^{(n)}}_{G : H' ; \dots ; H^{(n)}} \left(\begin{matrix} [(e) : \theta', \dots, \theta^{(n)}] : \\ [(g) : \gamma, \dots, \gamma^{(n)}] : \end{matrix} \right.$$
$$\left. \begin{matrix} [(f') : Q'] ; \dots ; [(f^{(n)}) : Q^{(n)}] ; \\ [(h') : \Omega'] ; \dots ; [(h^{(n)}) : \Omega^{(n)}] ; \end{matrix} Z_1 \sin^{2\rho_1} \frac{\pi x}{L}, \dots, Z_n \sin^{2\rho_n} \frac{\pi x}{L} \right).$$

By (7.3) , (7.4) and (7.5), it is clear that

$$(7.7) \qquad u(x, 0) = f(x) = \sum_{m=1}^{\infty} a_m \sin \frac{\pi \lambda_m x}{L}$$

and

$$(7.8) \qquad \frac{\partial u}{\partial t}(x, 0) = g(x) = \frac{\pi c}{L} \sum_{m=1}^{\infty} b_m \lambda_m \sin \frac{\lambda_m x}{L}$$

Now making an appeal to the integral (3.1), we find the values of a_m and b_m separately and put them in (7.5) to get required solution of the problem in the following form:

$$(7.9) \qquad u(x, t) = \frac{\pi}{2^{w-3}} \sum_{m=1}^{\infty} \frac{\cos \frac{\pi \lambda_m ct}{L} \sin \frac{\pi \lambda_m x}{L} \sin \frac{\pi \lambda_m}{2}}{(2 \pi \lambda_m - \sin 2 \pi \lambda_m)}$$

$$S^{A+1:B';\dots;B^{(n)}}_{C+2:D';\dots;D^{(n)}}\left(\begin{array}{l}[(a):\theta';\dots;\theta^{(n)}],\ [w:2\xi_1,\dots,2\xi_n]: \\ [(c):\psi',\dots,\psi^{(n)}],\left[\frac{w+1+\lambda_m}{2}:\xi_1,\dots,\xi_n\right],\end{array}\right.$$

$$\left.\begin{array}{l}[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ \left[\frac{w+1-\lambda_m}{2}:\xi_1,\dots,\xi_n\right]:(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{array} Z_1/4\xi_1,\dots,Z_n/4\xi_n\right)$$

$$+\frac{L}{2^{\omega'-3}.c}\sum_{m=1}^{\infty}\frac{\sin\frac{\pi\lambda_m}{2}\sin\frac{\pi\lambda_m ct}{L}\sin\frac{\pi x\lambda_m}{L}}{(2\pi\lambda_m-\sin 2\pi\lambda_m)}$$

$$S^{E+1:F';\dots;F^{(n)}}_{G+2:H';\dots;H^{(n)}}\left(\begin{array}{l}[(e):\theta',\dots,\theta^{(n)}], \\ [(g):\gamma',\dots,\gamma^{(n)}],\end{array}\right.$$

$$[\omega':2\rho_1,\dots,2\rho_n]:$$

$$\left[\frac{\omega'+1+\lambda_m}{2}:\rho_1,\dots,\rho_n\right],\left[\frac{\omega'+1-\lambda_m}{2}:\rho_1,\dots,\rho_n\right]:$$

$$\left.\begin{array}{l}[(f'):\theta'];\dots;[(f^{(n)}:Q^{(n)}]; \\ [(h'):\Omega'];\dots;[h^{(n)}):\Omega^{(n)}];\end{array} Z_1/4\rho_1,\dots,Z_n/4\rho_n\right)$$

where all $Re\,(\omega)>0$,

$$1+\sum_{j=1}^{C}\Psi_j^{i}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)}>0,$$

$$1+\sum_{j=1}^{G}\gamma_j^{(i)}+\sum_{j=1}^{H^{(i)}}\Omega_j^{(i)}-\underset{j=1}{\overset{E}{E}}\theta_j^{(i)}-\sum_{j=1}^{F}Q_j^{(i)}>0,$$

where all ξ_ι, ρ_i are real positive integers, $i=1,\dots,n$

8. SPECIAL CASE. For initial velocity

(8.1) $\frac{\partial u}{\partial t}(x,0)=g(x)=0,$

all b's in (7.5) will be zero. Thus our problem 2 now reduces to the problem
1. Therefore, making an apeal to the integral (3.1), solution of the problem is given by

(8.2) $$u(x,t)=\frac{\pi}{2^{\omega-3}}\sum_{m=1}^{\infty}\frac{\lambda_m\cos\left(\frac{\pi\lambda_m ct}{L}\right)\sin\frac{\pi\lambda_m x}{2}\sin\frac{\pi\lambda_m}{2}}{[2\pi\lambda_m-\sin 2\pi\lambda_m]}$$

$$S^{A+1:B';\dots;B^{(n)}}_{C+2:D';\dots;D^{(n)}}\left(\begin{array}{l}[(a):\theta',\dots,\theta^{(n)}],\ [\omega:2\xi_1,\dots,2\xi_n \\ [(c):\Psi',\dots,\Psi^{(n)}],\left[\frac{\omega+1+\lambda_m}{2}:\xi_1,\dots,\xi_n\right],\end{array}\right.$$

$$\left.\begin{array}{l} [(b') : \phi']; \ldots; [(b^{(n)}) : \phi^{(n)}]; \\ \left[\dfrac{\omega + 1 - \lambda_m}{2} : \xi_1, \ldots, \xi_n\right]; \ [(d') : \delta'] \ ; \ldots ; [(d^{(n)}) : \delta^{(n)}] \ ; \quad Z_1/4\,\xi_1, \ldots, Z_n/4\,\xi_n \end{array}\right)$$

where all conditions of (3.1) are satisfied.

REFERENCES

[1] R.C. Singh Chandel and H.C. Yadava, Heat conduction and the multiple hypergeometric function of Srivastava and Daoust, *Indian J.Pure Appl. Math.* **15** (1984), 371-376.

[2] R.C. Singh Chandel and A.K. Gupta, A Problem on heat conduction in a finite bar, *Journal of M.A.C.T.,* **19** (1986), 91- 95.

[3] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products,* Academic Press, Inc., New York, 1980.

[4] A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, Inc., New York, 1949.

(5] H.M. Srivastava and M.C.Daoust, On Eulerian integrals associated with Kampe Feriet's function, *Publ. Inst. Math (Beogard) (N.S)*, **9 (23)** (1969), 199-202.

[6] H.M. Srivastava and M.C. Daoust, Certain generalized Neumann expansions associated with the Kampe de Feriet function, *Nederl. Acad. Wetensch. Proc. Ser. A* **72** = *Indag. Math.* **31** (1969), 449- 457.

[7] H.M. Srivastava and M.C. Daoust, A note on the convergence of Kampe de Feriet's double hypergeometric series, *Math. Nachr.,* **53** (1972), 151-159.

[8] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, John Wiley and Sons, New York, 1985.

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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

MULTIVARIABLE ANALOGUES OF A CLASS OF POLYNOMIALS $T_n^{(a,k)}(x, r, p)$

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ABSTRACT

In the present paper, we introduce a multivariable analogue of the generalized class of polynomials $T_n^{(a,k)}(x, r, p)$ of Chandel through the Rodrigues' formula (1.4). Its further generalization is also discussed by Rodrigues' formula (6.1). Our polynomials may also very well be regarded as multivariable analogues of Lagurre, Hermite, Bessel polynomials and Truesdell polynomials (Erdelyi [9], Toscano [16], generalized Truesdell polynomials of Singh [11]. Chak's polynomials [5], generalized Laguerre polynomials of Singh-Srivastava [12] (see also Chatterjea [8]), Chatterjea's generalized Bessel polynomials [7] and generalized Hermite function of Gould-Hopper [10].

1. Introduction.

Srivastava and Singhal [15] studied a class of polynomials defined by generalized Rodrigues' formula (see also [14,p.401, problem 34]) :

$$(1.1)\quad G_n^{(\alpha)}(x, r, p, k) = \frac{1}{n\,!}\, x^{-\alpha-kn} \exp(px^r)\left(x^{k+1}\frac{d}{dx}\right)^n \{x^{\alpha}\exp(-px^r)\},$$

where the parameters α, r, p, and k are arbitrary in general. Subsiquently, but independently, to introduce generalized Stirling numbers and polynomials Chandel ([1], [2], [3]) studied a slight variation of Srivastava - Singhal polynomials in the form :

$$(1.2)\quad T_n^{(\alpha,k)}(x, r, p) = x^{-\alpha}\exp(px^r)\,\Omega_x^n\,\{x^{\alpha}\exp(-px^r)\}$$

where, and in what follows, $\Omega_x = x^k \dfrac{d}{dx}$.

Recently, Chandel and Tiwari ([4], (1.7)) introduced a multivariable analogue of Hermite polynomials defined by Rodrigues' formula

$$(1.3)\quad H_{n_1,\ldots,n_m}^{(b\,;\,h_1,\ldots,h_m;\,r_1,\ldots,r_m)}(x_1, \ldots, x_m)$$

$$= (-1)^{n_1 + \dots + n_m} (1 + h_1 x_1^{r_1} + \dots + h_m x_m^{r_m})^b$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \cdots \frac{d^{n_m}}{dx_m^{n_m}} (1 + h_1 x_1^{r_1} + \dots + h_m x_m^{r_m})^{-b},$$

where $n_1, \dots, n_m$ are positive integers while $h_1, \dots, h_m$; $r_1, \dots, r_m$ and b are any numbers real or complex independent of variables $x_1, \dots, x_m$.

In the present paper, first we introduce the multivariable analogue

$T^{(a_1, \dots, a_m; h_1, \dots, k_m; r_1, \dots, r_m; p_1, \dots, p_m; b)}_{n_1, \dots, n_m} (x_1, \dots, x_m)$

of the class of polynomials of Chandel ([1], [2], [3]), defined by

$$(1.4)\ T^{(a_1, \dots, a_m; k_1, \dots, k_m; r_1, \dots, r_m; p_1, \dots, p_m; b)}_{n_1, \dots, n_m} (x_1, \dots, x_m)$$

$$= x_1^{-a_1} \dots x_m^{-a_m} (1 + p_1 x_1^{r_1} + \dots + p_m x_m^{r_m})^b$$

$$\Omega_{x_1}^{n_1} \dots \Omega_{x_m}^{n_m} \left\{ x_1^{a_1} \dots x_m^{a_m} \left(1 + p_1 x_1^{r_1} + \dots + p_m x_m^{r_m} \right)^{-b} \right\}, \Omega_{x_i} = x^{k_i} \frac{\partial}{\partial x_i},$$

where $a_1, \dots, a_m, k_1, \dots, k_m, r_1, \dots, r_m, p_1, \dots, p_m, b$ are arbitrary in general independent of variables $x_1, \dots, x_m$ but no $k_i = 1; i = 1, \dots, n$. Finally, we shall discuss further generalization of (1.4) through Rodrigues' formula (6.1).

It is clear that

$$(1.5)\ \lim_{b \to \infty} T^{(a_1, \dots, a_m; k_1, \dots, k_m; r_1, \dots, r_m; p_1/b, \dots, p_m/b; b)}_{n_1, \dots, n_m} (x_1, \dots, x_m)$$

$$= T^{(a_1, k_1)}_{n_1} (x_1, r_1, p_1) \dots T^{(a_m, k_m)}_{n_m} (x_m, r_m, p_m),$$

where $T^{(a, k)}_n (x, r, p)$ are polynomials of Chandel ([1], [2], [3]) defined by (1.2).

For $k_1 = \dots = k_m = a_1 = \dots a_m = 0$, (1.4) reduces to (1.3), showing

$$(1.6)\ T^{(0, \dots, 0; 0, \dots, 0; r_1, \dots, r_m; p_1, \dots, p_m; b)}_{n_1, \dots, n_m} (x_1, \dots x_m)$$

$$= (-1)^{n_1 + \dots + n_m} H^{(b; p_1, \dots, p_m; r_1, \dots, r_m)}_{n_1, \dots, n_m} (x_1, \dots, x_m).$$

For $k_1 = \dots k_m = 0$, (1.4) reduces to the polynomials of Chandel and Tiwari [5, p. 760 (5.1)] defined by

$$(1.7)\ H^{(a_1, \dots, a_m; r_1, \dots, r_m; p_1, \dots, p_m; b)}_{n_1, \dots, n_m} (x_1, \dots, x_m)$$

$$= (-1)^{n_1 + \dots + n_m} x_1^{-a_1} \dots x_m^{-a_m} [1 + p_1 x_1^{r_1} + \dots + p_m x_m^{r_m}]^b$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \cdots \frac{d^{n_m}}{dx_m^{n_m}} \left\{ x_1^{a_1} \ldots x_m^{a_m} (1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})^{-b} \right\},$$

as special case of their polynomials [5, p. 757 (1.1)].

Choosing $k_1 = \ldots = k_m = 0$ and replacing p_i by p_i/b $(i = 1, \ldots, m)$ and then taking $b \to \infty$, (1.4) reduces to the polynomials of Chandel and Tiwari [5, p. 760 (5.2)] defined by

$$(1.8)\ E_{n_1, \ldots, n_m}^{(a_1, \ldots, a_m; r_1, \ldots, r_m; p_1, \ldots, p_m)} (x_1, \ldots x_m)$$

$$= (-1)^{n_1 + \ldots + n_m} x_1^{-a_1} \ldots x_m^{-a_m} \exp[-(p_1 x_1^{r_1} + \ldots p_m x_m^{r_m})]$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \cdots \frac{d^{n_m}}{dx_m^{n_m}} \left\{ x_1^{a_1} \ldots x_m^{a_m} \exp(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}) \right\},$$

as special case of their polynomials [5, p.557 (1.1)].

Our polynomials may also very well be regarded as the multivariable analogue of Laguree, Hermite, Bessel polynomials and the Truesdell polynomials (Erdelyi [9], Toscano [16]), generalized Truesdell polynomials of Singh [11], Chak's polynomials [6], Singh and Srivastava generalized Laguerre polynomials [12] (see also Chatterjea [8]), Chatterjea's generalized Bessel polynomials [7] and generalized Hermite function of Gould and Hopper [10].

For brevity, we shall write

$$T_{n_1, \ldots, n_m}^{(a_1, \ldots, a_m; k_1, \ldots, k_m; r_1, \ldots, r_m; p_1, \ldots, p_m; b)} (x_1, \ldots, x_m)$$

as

$$T_{n_1, \ldots, n_m}^{(a_1, \ldots, a_m; [k_m]; [r_m]; [p_m]; b)} (x_1, \ldots, x_m)$$

2. **Generating Relation.** Starting with the Rodrigues' formula (1.4) and making an appeal to the result due to Chandel [1, p.43, (2.5)]

$$(2.1)\ e^{t\Omega_x} \{f(x)\} = f(x\{1 - (k-1)t\, x^{k-1}\}^{-1/(k-1)}),$$

we derive the following generating relation :

$$(2.2) \qquad \sum_{n_1, \ldots, n_m = 0}^{\infty} T_{n_1, \ldots, n_m}^{(a_1, \ldots, a_m; [k_m]; [r_m]; [p_m]; b)} (x_1, \ldots, x_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$= (1 + p x_1^{r_1} + \ldots + p_m x_m^{r_m})^b\, [1 - (k_1 - 1) t_1 x_1^{k_1 - 1}]^{-a_1/(k_1 - 1)}$$

$$\ldots [1 - (k_m - 1) t_m\, x_m^{k_m - 1}]^{-a_m/(k_m - 1)}$$

$$\left[1 + \frac{p_1 x_1^{r_1}}{[1 - (k_1 - 1) t_1 x_1^{k_1 - 1}]^{r_1/(k_1 - 1)}} + \right.$$

$$\ldots + \left.\frac{p_m x_m^{r_m}}{[1-(k_m-1)t_m\, x_m^{k_m-1}]^{r_m/(k_m-1)}}\right]^{-b} :$$

3. Explicit form. Starting with the generating relation (2.2), we derive

$$\text{(3.1)}\quad T_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$= (1+p_1x_1^{r_1}+\ldots p_m x_m^{r_m})^b\,(k_1-1)^{n_1}\ldots(k_m-1)^{n_m} x_1^{n_1(k_1-1)}\ldots$$

$$x_m^{n_m(k_m-1)}\left(\frac{a_1}{k_1-1}\right)_{n_1}\cdots\left(\frac{a_m}{k_m-1}\right)_{n_m}$$

$$\sum_{s_1,\ldots,s_m=0}^{\infty}\frac{(b)_{s_1\ldots+\ldots s_m}\left(\frac{a_1}{k_1-1}+n_1\right)_{r_1s_1/(k_1-1)}\cdots\left(\frac{a_m}{k_{m-1}}+n_m\right)_{r_ms_m/(k_m-1)}}{\left(\frac{a_1}{k_1-1}\right)_{r_1s_1/(k_1-1)}\cdots\left(\frac{a_m}{k_m-1}\right)_{r_ms_m/(k_m-1)}}$$

$$\frac{(-p_1x_1^{r_1})^{s_1}}{s_1\,!}\cdots\frac{(-p_m\,x_m^{r_m})^{s_m}}{s_m\,!},$$

which can be written in the following form of multiple hypergeometric function of Srivastava and Daoust [13] :

$$\text{(3.2)}\quad T_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$= (k_1-1)^{n_1}\ldots(k_m-)^{n_m}\,x_1^{n_1(k_1-1)}\ldots x_m^{n_m(k_m-1)}\left(\frac{a_1}{k_1-1}\right)_{n_1}\cdots\left(\frac{a_m}{k_m-1}\right)_{n_m}$$

$$(1+p_1x_1^{r_1}+\ldots+p_m\,x_m^{r_m})^b$$

$$F_{0:1;\ldots;1}^{1:1;\ldots;1}\left(\begin{array}{l}[b:1,\ldots,1]:\left[\frac{a_1}{k_1-1}+n_1:\frac{r_1}{k_1-1}\right];\ldots;\left[\frac{a_m}{k_m-1}+n_m:\frac{r_m}{k_m-1}\right];\\ \qquad -:\left[\frac{a_1}{k_1-1}:\frac{r_1}{k_1-1}\right];\ldots;\left[\frac{a_m}{k_m-1}:\frac{r_m}{k_m-1}\right];\\ \qquad\qquad -p_1\,x_1^{r_1},\ldots,-p_m\,x_m^{r_m}\end{array}\right).$$

4. Applications of Generating Relation. Making an appeal to the generating relation (2.2), we derive

$$(4.1)\ T_{n_1,\ldots,n_m}^{(a_1+c_1,\ldots,a_m+c_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b+d)}(x_1,\ldots,x_m)$$

$$=\sum_{s_1=0}^{n_1}\cdots\sum_{s_m=0}^{n_m}\binom{n_1}{s_1}\cdots\binom{n_m}{s_m}T_{n_1-s_1,\ldots,n_m-s_m}^{(a_1,\ldots a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$T_{s_1,\ldots,s_m}^{(c_1,\ldots,c_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,d)}(x_1,\ldots,x_m).$$

$$(4.2)\ T_{n_1,\ldots,n_m}^{(a_1+c_1,\ldots,a_m+c_m\,;\,[k_m]\,;\,[r_m]\,;\,;\,[p_1\,;\,b)}(x_1,\ldots,x_m)$$

$$=\sum_{s_1=0}^{n_1}\cdots\sum_{s_m=0}^{n_m}\cdots\binom{n_1}{s_1}\cdots\binom{n_m}{s_m}(k_1-1)^{s_1}\cdots(k_m-1)^{s_m}\left(\frac{a_1}{k_1-1}\right)_{s_1}\cdots\left(\frac{a_m}{k_m-1}\right)_{s_m}$$

$$x_1^{s_1(k_1-1)}\cdots x_m^{s_m(k_m-1)}\,T_{n_1-s_1,\ldots,n_m-s_m}^{(c_1,\ldots,c_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots x_m),$$

and

$$(4.3)\ T_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b+d)}(x_1,\ldots,x_m)$$

$$=(1+p_1x_1^{r_1}+\ldots+p_m x_m^{r_m})^b\sum_{s_1,\ldots,s_m=0}^{\infty}(b)_{s_m+\ldots+s_1}\frac{(-p_1x_1^{r_1})^{s_1}}{s_1!}\cdots\frac{(-p_m x_m^{r_m})^{s_m}}{s_m!}$$

$$T_{n_1,\ldots,n_m}^{(a_1+r_1s_1,\ldots,a_m+r_ms_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,d)}(x_1,\ldots x_m).$$

5. Recurrence Relations. Starting with generating relation (2.2). we derive

$$(5.1)\ T_{n_1,\ldots,n_m}^{(a_1+k_1-1,\,a_2,\ldots,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots x_m)$$

$$+(k_1-1)x_1^{k_1-1}\,T_{n_1-1,n_2,\ldots,n_m}^{(a_1+k_1-1,\,a_2,\ldots a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$=T_{n_1,\ldots n_m}^{(a_1,\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots x_m),$$

which suggests to unify m- recurrence relations in the foolowing form :

$$(5.2)\ T_{n_1,\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i+k_i-1,\,a_{i+1},\ldots,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$+(k_i-1)x_i^{k_i-1}\,n_i\,T_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i+k_i-1,\,a_{i+1},\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$=T_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m),$$

$i=1,\ldots,m.$

An appeal to generating relation (2.2) also gives

(5.3) $(1+p_1 x_1^{r_1}+\ldots+p_m x_m^{r_m})\, T_{n_1,\ldots n_m}^{(a_1,\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b-1)}(x_1,\ldots,x_m)$

$$= T_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$+\sum_{i=1}^{m} p_i x_i^{r_i}\, T_{n_1,\ldots n_m}^{(a_1,\ldots,a_{i-1},\,a_i+r_i,\,a_{i+1},\,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

and

(5.4) $T_{n_1+1,\,n_2,\ldots,n_m}^{(a_1,\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$

$$= a_1 x_1^{k_1-1}\, T_{n_1,\ldots,n_m}^{(a_1+k_1-1,\,a_2,\ldots,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$$

$$- b p_1 r_1 x_1^{r_1+k_1-1}\, T_{n_1,\ldots,n_m}^{(a_1+r_1+k_1-1,\,a_2,\ldots,a_m\,;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b+1)}(x_1,\ldots,x_m),$$

which further suggests to unify m- similar results in the form :

(5.5) $T_{n_1,\ldots,n_{i-1},\,n_i+1,\,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_1,\ldots,x_m)$

$$= a_i x_i^{k_i-1}\, T_{n_1,\ldots,n_m}^{(a_1,\ldots a_{i-1},\,a_i+k_i-1,\,a_{i+1},\ldots,a_m;\,[k_m]\,;\,[r_m]\,;\,[p_m]\,;\,b)}(x_{,}\ldots,x_m)$$

$$- b p_i r_i x_i^{r_i+k_i-1}\, T_{n_1,\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i+r_i+k_i-1,\,a_{i+1},\ldots,a_m\,;\,[k_m];\,[r_m]\,;\,[p_m]\,;\,b+1)}(x_1,\ldots,x_m),$$

$i = 1, \ldots, m.$

6. Generalization. Consider

(6.1) $G_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,[k_m];\,[r_m];\,[p_m])}(x_1,\ldots,x_m)$

$$= x_1^{-a_1}\ldots x_m^{-a_m}\,[G(p_1 x_1^{r_1}+\ldots+p_m x_m^{r_m})]^{-1}$$

$$\Omega_{x_1}^{n_1}\ldots\Omega_{x_m}^{n_m}\,\{x_1^{a_1}\ldots x_m^{a_m}\,G\,(p_1 x_1^{r_1}+p_m x_m^{r_m})\},$$

where

(6.2) $G(z)=\sum_{n=0}^{\infty}\gamma_n x^n,\ r_0\neq 0.$

It is clear that for $\gamma_n=\frac{(-1)^n (b)_n}{n\,!}$, (6.1) reduces to (1.4), while for $\gamma_n=\frac{(-1)^n}{n\,!}$, (6.1) gives

(6.3) $E_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,[k_m]\,;\,[r_m];\,[p_m])}(x_1,\ldots,x_m)$

$$= T_{n_1}^{(a_1,\,k_1)}(x_1, r_1, p_1)\ldots T_{n_m}^{(a_m,\,k_m)}(x_m, r_m, p_m).$$

7. **Generating Relation.** Starting with the Rodrigues' formula (6.1) and making a appeal to (2.1) , we drive the following generating relation :

$$(7.1)\quad \sum_{n_1, \dots, n_m = 0}^{\infty} G_{n_1, \dots, n_m}^{(a_1, \dots, a_m ; [k_m] ; [r_m]; [p_m])} (x_1, \dots, x_m) \frac{t_1^{n_1}}{n_1 !} \dots \frac{t_m^{n_m}}{n_m !}$$

$$= \frac{[1-(k_1-1)\, t_1\, x^{k_1-1}]^{-a_1/(k_1-1)} \dots [1-(k_m-1)\, t_m\, x_m^{k_m-1}]^{-a_m/(k_m-1)}}{G(p_1\, x_1^{r_1} + \dots p_m\, x_m^{r_m})}$$

$$G\left(\frac{p_1 x_1^{r_1}}{[1-(k_1-1)t_1\, x_1^{k_1-1}]^{r_1/(k_1-1)}} + \dots + \frac{p_m\, x_m^{r_m}}{[1-(k_m-1)\, t_m\, x_m^{k_m-1}]^{\, r_m/k_m-1)}}\right).$$

8. **Explicit Form**. Starting with the generating relation (7.1), we derive the following explicit form :

$$(8.1)\quad G_{n_1, \dots, n_m}^{(a_1, \dots a_m ; [k_m] ; [r_m] ; [p_m])} (x_1, \dots, x_m)$$

$$= \frac{\left(\frac{a_1}{k_1-1}\right)_{n_1} \dots \left(\frac{a_m}{k_m-1}\right)_{n_m} (k_1-1)^{n_1} x_1^{(k_1-1)n_1}, \dots (k_m-1)^{n_m} x_m^{(k_m-1)n_m}}{G\,(p_1\, x_1^{r_1} + \dots + p_m\, x_m\, x_m^{r_m})}.$$

$$\sum_{s_1, \dots, s_m = 0}^{\infty} \frac{\gamma_{s_1 + \dots + s_m} \left(\frac{a_1 + (k_1-1)n_1}{k_1-1}\right)_{r_1 s_1/(k_1-1)} \dots \left(\frac{a_m + (k_m-1)n_m}{k_m-1}\right)_{r_m s_m/(k_m-1)}}{\left(\frac{a_1}{k_1-1}\right)_{r_1 s_1/(k_1-1)} \dots \left(\frac{a_m}{k_m-1}\right)_{r_m s_m/(k_m-1)}}$$

$$\cdot \frac{(p_1\, x_1^{r_1})^{s_1}}{s_1 !} \dots \frac{(p_m\, x_m^{r_m})^{s_m}}{s_m !}.$$

9. **Recurrence Relations.** Making an appeal to generating relation (7.1), we derive

$$(9.1)\quad G_{n_1, \dots, n_m}^{(a_1, \dots, a_m ; [k_m] ; [r_m] ; [p_m])} (x_1, \dots, x_m)$$

$$= G_{n_1, \dots, n_m}^{(a_1 + k_1 - 1, a_2, \dots, a_m; [k_m] ; [r_m]; [p_m])} (x_1, \dots, x_m)$$

$$- (k_1 - 1)\, n_1\, x_1^{k_1-1}\, G_{n_1-1, n_2, \dots, n_m}^{(a_1 + k_1 - 1, a_2, \dots, a_m ; [k_m]; [r_m]; [p_m])} (x_1, \dots, x_m),$$

which suggests that *m*-results similar to (9.1) can be unified in the form :

$$(9.2)\; G_{n_1, \ldots, n_m}^{(a_1, \ldots, a_m; [k_m]; [r_m]; [p_m])}(x_1, \ldots, x_m)$$

$$= G_{n_1, \ldots, n_m}^{(a_1, \ldots, a_{i-1}, a_i + k_i - 1, a_{i+1}, \ldots, a_m; [k_m]; (r_m]; [p_m])}(x_1, \ldots, x_m)$$

$$- (k_i - 1)\, n_i\, x_i^{k_i - 1}$$

$$G_{n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_m}^{(a_1, \ldots, a_{i-1}, a_i + k_i - 1, a_{i+1}, \ldots, a_m; [k_m]; [r_m]; [p_m])}(x_1, \ldots, x_m),$$

$$i = 1, \ldots, m.$$

Also

(9.3)

$$p_j\, r_j\, x_j^{r_j + k_j}\, G_{n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots, n_m}^{(a_1, \ldots, a_{j-1}, a_j + r_j + k_j - 1, a_{J+1}, \ldots, a_m; [k_m]; [r_m]; [p_m])}(x_1, \ldots, x_m)$$

$$- a_i\, p_j\, r_j\, x_i^{k_i - 1}\, x_j^{r_j + k_j} \times$$

$$G_{n_1, \ldots, n_m}^{(a_1, \ldots, a_{i-1}, a_i + k_i - k, a_{i+1}, \ldots, a_{j-1}, a_j + r_j + k_j - 1, a_{j+1}, \ldots, a_m; [k_m]; [r_m]; [p_m])}(x_1, \ldots, x_m)$$

$$= p_i\, r_i\, x_i^{r_i + k_i}\, G_{n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_m}^{(a_1, \ldots, a_{i-1}, a_i + r_i + k_i - 1, a_{i+1}, \ldots a_m; [k_m]; [r_m]; [p_m])}(x_1, \ldots, x_m)$$

$$- p_i\, r_i\, a_j\, x_j^{k_j - 1}\, x_i^{r_i + k_i}\,.$$

$$G_{n_1, \ldots, n_m}^{(a_1, \ldots, a_{i-1}, a_i + r_i + k_i - 1, a_{i+1}, \ldots, a_{j-1}, a_j + k_j - 1, a_{j+1} \ldots, a_m; [k_m]; [r_m]; [p_m])}(x_1, \ldots, x_m),$$

where $i, j \in \{1, \ldots, m\}, i \neq i$.

REFERENCES

[1] R.C.S. Chendel, A new class of polynomials, *Indian J. Math.*, 15 (1973), 41-49.

[2] R.C.S. Chandel, A further note on the polynomials $T_n^{(\alpha, h)}(x, r, p)$, *Indian J. Maith.*, 16 (1974), 39-48.

[3] R.C.S. Chandel, Generalized Stirling numbers and polynomials, *Publ. del Institute Mathematique*, 22 (36) (1977), 43-48.

[4] R.C.S. Chandel and A. Tiwari, Multivariable analogue of Hermite polynomials, *Ganita Sandesh*, 5 (1991), 92-95.

[5] R.C.S. Chandel and A. Tiwari, Multivariable analogue of Gould and Hopper's polynomials defined by Rodrigues' formula, *Indian J. Pure Appl. Math.* 22 (9) (1991), 757-761.

[6] A.M. Chak, A class of polynomials and generalization of Stirling numbers, *Duke Math.* Jour, 23 (1956), 45-55.

[7] S.K. Chatterjea, A generalization of Bessel polynomials, *Mathematica*, 6 (1964), 19-29.

[8] S.K. Chatterjea, On a generalization of Laguree polynomials, Rendiconti del *Seminario Matamatico della Univ. di Padova*, 34 (1964), 180-190.

[9] A. Erdelyi et al, *Higher Transcendental Functions*, 3, New York, 1955, p. 254.

[10] H.W. Gould and A.T. Hopper, Operational formulas connected with two

generalizations of Hermite polynomials, *Duke Math. Jour.*, 29 (1962), 55-64.

[11] R.P. Singh, On generalized Triuesdell polynomials, *Rivista di Matematica* (1968).

[12] R.P. Singh and K.N. Srivastava, A. note on generalizations of Laguerre and Hermite polynomials, *Ricerca (Napoli)* (2), 14 (1963), settembre-decembre, 11-21' errata, ibid (2), 15 (1964), maggioangusto, 63.

[13] . H.M. Srivastava and M.C. Daoust, Certain generalized Neumann expansions associated with the Kampe de Feriet function, *Nederl. Akad. Wetensch. Proc. Ser.* A 72 = Indag. Math 31 (1969), 449-457.

[14] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester) Hohn Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[15] H.M. Srivastava and J.P. Singhal, A class of polynomials defined by generalized Rodrigues' formula, *Ann. Mat. Pura Appl.* (4) 90 (1971), 75-85.

[16] L. Toscano, Una class di polinomi della matematica atturaiale *Rivista di Matematica della Universita di Parma*, 1 (1950), 459- 470.

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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

FIXED POINTS OF WEAKLY COMMUTING MAPPINGS ON MENGER SPACES

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ABSTRACT

In this note we prove fixed point theorems for four mappings on a Menger space. These results are good extensions and improvements of Stojakovic [23] and several other known results on metric and Menger spaces.

INTRODUCTION

Sehgal and Bharucha-Reid [16] initiated the study of fixed points in a subclass of probabilistic metric spaces. Subsequently, there has been a steady growth of fixed point theorems in probabilistic metric and Menger spaces (see, for instance, [1], [3], [8], [13], [14], [19]- [21], [23], [24], [27] and a good bibliography may be found in [21].In fact, fixed point theorems for certain contractive type mappings on Menger spaces with proper choice of the "norm" may be obtained from corresponding results in metric ~~spaces~~ (see, [8] and [14]).

On the other hand, the potential of commuting mappings (see,jungck [10]) was used by various authors to present a number of fixed point theorems in metric spaces by generalizing the spatial structure or the constraints on the mappings. Sessa [17] (see also [11]-[13], [18], [21]) relaxed the concept of commuting to that of weakly commuting mappings (cf. Def.1) and observed that commuting mappings are weakly commuting but not conversely, Recently,two less restrictive concepts than of weak commutativity, in independent formulations [11] and [25],were introduced under the names of compatible mappings and asymptotically commuting mappings.

In this paper we prove common fixed point theorems for four mappings under the condition of weak commutativity. We take only one of the mappings continuous. Our results extend and improve certain results, among others, of [2]-[7], [9], [11], [12], [18], [23], [26] and [27]. The method of proof suggests that our results are valid for asymptotically commuting or compatible mappings.

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PRELIMINARIES

A probabilistic metric space (PM_space) is an ordered pair $(X, \mathcal{F})$ consisting of a nonempty set X and a mapping $\mathcal{F}$ from $X \times X$ to L, the collection of all distribution functions. The value of t at $(u, v) \in X \times X$ is generally represented by $F_{u,v}$ or, for the sake of convenience by $F(u,v)$. The functions $F(u,v)$ are assumed to satisfy the following conditions:-

(a) $F(u, v; x) = 1$ for all $x > o, u = v$;

(b) $F(u, v, 0) = 0$ for all u, v in X ;

(c) $F(u, v) = F(v, u)$ for all u, v in X;

(d) If $F(u, v; x) = 1$, $F(v, w; y) = 1$ then $F(u, w; x + y) = 1$ for all $u, v; w$ in X and $x, y > 0$.

A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM-space and t is t-norm [15] such that the inequality:

$F(u, w; x + y) \geq t \{F(u, v; x), F(v, w; y)\}$ holds for all $u, v, w \in X$ and all $x \geq 0, y \geq 0$.

For Topological preliminaries on a Menger space, Schweizer and Sklar [15] makes an interesting reading.

Definition 1 [17]. Two self-mappings g and h of a metric space (M,d) are weakly commuting iff

$$d(ghu, hgu) \leq d(hu, gu) \text{ for all } u \in M .$$

Defintion 2. Two self-mappings g and h of a PM-space X afe weakly commuting iff

$$F(ghu, hgu; x) \geq F(hu, gu; x) \text{ for all } u \in X \text{ and } x > 0.$$

Definition 3 [11]-[12]. Two Self-mappings g and h of a metric space (M, d) are compatible (also called y-asymptotically commuting [25]) iff $\lim_n d(ghu_n, hgu_n) = 0$ when $\{u_n\}$ is a sequence such that $\lim_n hu_n = \lim_n gu_n = y$ for some y in X.

Definition 4 [13]. Two self-mappings g and h of a PM-space X are compatible (also called y-asymptotically commuting) iff $F(ghx_n, hgx_n; x) \longrightarrow 1$ for all $x > 0$ whenever $\{x_n\}$ is a sequence in X such that $gx_n, hx_n \longrightarrow y$ for some y in X

Clearly, if g and h commute or weakly commute then they are compatible but the converse is not true (see [11], [12], [20]).

EXAMPLES

Example 1. Let $M = [1, \infty]$ with the absolute value metric d and let g and h be two mappings on M defined by $gu = 1+2u$ and $hu = 1+4u$.

Then $d(ghu, hgu) = 2 \leq 2u = d(hu, gu)$. Hence the mappings are weakly commuting. Note that g and h do not commute.

In the above example there exists no sequence $\{u_n\}$ in M for which the condition of cmpatibility (or asymptotic commutativity) may be satisfied.Thus, weakly commuting mappings mappings may be vacuously compatible (or asymptotically commuting).

Example 2. Let $X = \{p, q, r\}$ and let $\mathcal{F}$ be defined via

$$F(p, r; x) = F(r, p; x) = F(r, q; x) = F(q, r; x)$$

$$= \begin{cases} 0, \text{ if } x \leq 0, \\ \frac{1}{2}, \text{ if } 0 < x \leq 2, \\ 1, \text{ if } x > 2 . \end{cases}$$

and

$$F(p, q; x) = F(q, p; x) = \begin{cases} 0, \text{ if } x \leq 0, \\ \frac{1}{2}, \text{ if } 0 < x \leq 3/2, \\ 1, \text{ if } x > 3/2 . \end{cases}$$

Then $(X, \mathcal{F}, t_m)$ is a Menger space [14, Ex.3.2]. Let f,g: $X \longrightarrow X$ be such that

$f(p) = f(q) = p,\ f(r) = q$

and $g(p) = g(q) = g(r) = p$. Then it can be verified that f and g are weakly commuting but not commuting.

FIXED POINT THEOREMS

Theorem 1. Let $(X, \mathcal{F}, t)$ be a complete Menger space where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$ and mappings $h, k : X \longrightarrow X$, $f: X \longrightarrow h(X)$ and $g : X \to k(X)$ be such that f and g weakly commute with k and h respectively and one of f, g, h, k, be continuous.

Further, let for all $u, v \in X$ and $x \geq 0$

(1) $F(fu, gv: x) \geq t\left\{F(ku, hv; \phi(x)), t\left\{F(fu, ku; \phi(x), F(gv, hv; (\phi(x))\right\}\right\}$

hold, where $\phi : R^+ \longrightarrow R^+$ is an increasing function such that $\lim_n \phi^n(x) = \infty$ for all $x > 0$. Then f, g, h and k have a unique common fixed point. Indeed, the pairs (f, k) and (g, h) have unique common fixed points and these two points coincide.

Proof. For $u_0 \in X$, we choose a sequence $\{u_n\}$ in X such that

$$fu_{2n} = hu_{2n+1} = y_{2n} \text{ (say), and}$$

$$gu_{2n-1} = ku_{2n} = y_{2n-1} \text{ (say).}$$

This is valid since $f: X \longrightarrow h(X)$ and $g: X \longrightarrow k(X)$

By (1), for $x > 0$

$$F(y_{2n}, y_{2n+1}; x) \doteq F(fu_{2n}, gu_{2n+1}; x)$$

$$\geq t\Big\{F(y_{2n-1}, y_{2n}; \phi(x)), t\Big\{F(y_{2n}, y_{2n-1}; \phi(x)), F(y_{2n+1}, y_{2n}; \phi(x))\Big\}\Big\}.$$

Using the associativity of t and $t(x, x) \geq x$,

$$F(y_{2n}, y_{2n+1}; x) \geq t\Big\{F(y_{2n}, y_{2n-1}; \phi(x)), F(y_{2n}, y_{2n+1}; \phi(x))\Big\}$$

$$\geq t\Big\{F(y_{2n}, y_{2n-1}; \phi(x)), t\Big\{F(y_{2n}, y_{2n-1}; \phi^2(x)), F(y_{2n}, y_{2n+1}; \phi^2(x))\Big\}\Big\}$$

$$\geq t\Big\{F(y_{2n}, y_{2n-1}; \phi(x)), F(y_{2n}, y_{2n+1}; \phi^2(x))\Big\}$$

$$\geq \quad \cdots\cdots \qquad \cdots\cdots$$

$$\geq t\Big\{F(y_{2n}, y_{2n-1}; \phi(x), F(y_{2n}, y_{2n+1}; \phi^2(x))\Big\}$$

Since $F(y_{2n}, y_{2n+1}; \phi^r(x)) \to 1$ *as* $r \longrightarrow \infty$, we get

$$F(y_{2n} y_{2n+1}; x) \geq F(y_{2n-1}, y_{2n}; \phi(x)).$$

Similarly

$$F(y_{2n+1}, y_{2n+2}; x) \geq F(y_{2n}, y_{2n+1}; \phi(x)).$$

Thus, in general

(2) $F(y_n, y_{n+1}; x) \geq F(y_{n-1}, y_n; \phi(x))$.

Now for $\varepsilon > 0$ and positive integers m, n $(m > n)$, we have with the application of (1)

$$F(y_n, y_m; \phi(\varepsilon)) \geq t\Big\{F(y_n, y_{n+1}; (\phi(\varepsilon) - \varepsilon), F(y_{n+1}, y_m; \varepsilon)\Big\}$$

$$\geq t\Big\{F(y_0, y_1; \phi^n(\phi(\varepsilon) - \varepsilon)), F(y_{n+1}, y_m; \varepsilon)\Big\}$$

$$\geq t\Big\{F(y_0, y_1; \phi^n(\phi(\varepsilon) - \varepsilon)), t\Big\{F(y_{n+1}, y_{n+2}; (\phi(\varepsilon) - \varepsilon)), F(y_{n+2}, y_m; \varepsilon)\Big\}\Big\}$$

$$\geq t\Big\{f(y_0, y_1; \phi^n(\phi(\varepsilon) - \varepsilon), t\Big\{F(y_0, y_1; \phi^{n+1}(\phi(\varepsilon) - \varepsilon)), F(y_{n+2}, y_m; \varepsilon)\Big\}\Big\}$$

$$\geq t\Big\{\ldots t\Big\{\ldots t\Big\{\ldots F(y_0, y_1; \phi^{m-1}(\varepsilon))\Big\}\Big\}\Big\}$$

Thus $F(y_n, y_m; \phi(\varepsilon)) \to)$ *as* $n \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence and it converges to some point z (say) in X. The sequence $Fu_{2n} = hu_{2n+1} \to z$, $gu_{2n+1} = ku_{2n} \to z$.

Let k be continuous. Then $kfu_{2n} \to kz$ and $k^2 u_{2n} \to kz$. Since f and g commute weakly, for $\varepsilon > 0$ *we have*

$$F(fku_{2n}, kz; \varepsilon) \geq t\Big\{F(fku_{2n}, kfu_{2n}; \varepsilon/2), F(kfu_{2n}, kz; \varepsilon/2)\Big\}$$

$$\geq t\Big\{F(ku_{2n}, fu_{2n}; \varepsilon/2), F(kfu_{2n}, kz; \varepsilon/2)\Big\}$$

$$\geq 1 - \lambda \text{ as } kfu_{2n} \to kz \text{ and } ku_{2n}, fu_{2n} \to z.$$

Thus $fku_{2n} \to kz$.

Since $\left\{ k^2 u_{2n} \right\}, \left\{ fku_{2n} \right\}$ converge to kz and $\left\{ gu_{2n+1} \right\}, \left\{ hu_{2n+1} \right\}$ converge to z, for $\varepsilon, \lambda > 0$ there exists $N = N(\varepsilon, \lambda)$ such that

(3) $F(k^2u_{2n}, fku_{2n}; \phi(\varepsilon) - \varepsilon) > 1 - \lambda, F(gu_{2n+1}, hu_{2n+1}; \varepsilon/2 > 1 - \lambda$

and $F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon) > 1 - \lambda$ for $n \geq N$.

Now by (1),

$$F(fku_{2n}, z; \phi(\varepsilon)) \geq t\left\{F(fku_{2n}, gu_{2n+1}; \varepsilon), F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon)\right\}$$

$$\geq t\{t\{F(k^2u_{2n}, hu_{2n+1}; \phi(\varepsilon)), t\{F(fku_{2n}, k^2u_{2n}; \phi(\varepsilon)),$$
$$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon)), F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon$$

$$\geq t\{t\{F(k^2u_{2n}, fku_{2n}; \phi(\varepsilon) - \varepsilon), F(fku_{2n}, gu_{2n+1}; \varepsilon/2),$$
$$F(gu_{2n+1}, hu_{2n+1}; \varepsilon/2), t\{F(fku_{2n}, k^2u_{2n}; \phi(\varepsilon)),$$
$$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon)\}\}, F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon)\}$$

$$= t\{F(k^2u_{2n}, fku_{2n}; \phi(\varepsilon) - \varepsilon), F(gu_{2n+1}, hu_{2n+1}\, \varepsilon/2),$$
$$F(gu_{2n} + 1, z; \phi(\varepsilon) - \varepsilon)\}$$
$$> 1 - \lambda \text{ from (3)}.$$

Therefore $fku_{2n} \to z$, proving $kz = z$.

Again by (1),

$$F(fz, gu_{2n+1}; \varepsilon)$$
$$\geq t\{F(kz, hu_{2n+1}; \phi(\varepsilon)), t\{F(fz, kz; \phi(\varepsilon)),$$
$$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon))\}\}$$
$$= t\left\{F(z, hu_{2n+1}; \phi(\varepsilon)), t\left\{F(fz, z; \phi(\varepsilon)), F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon))\right\}\right\}$$
$$\geq t\{F(z, hu_{2n+1}; \phi(\varepsilon)), t\{F(fz, gu_{2n+1}; (\varepsilon)), F(gu_{2n+1}, z; \phi(\varepsilon) - \varepsilon),$$
$$F(gu_{2n+1}, hu_{2n+1}; \phi(\varepsilon))\}\}$$
$$> 1 - \lambda, \text{ by (3)},$$

This implies $fz = z = kz$. Since $f : X \to h(X)$, there exists a z' in X such that $fz = z = hz'$. So, by (1),

$$F(z, gz'; x) = f(fz, gz'; x)$$
$$\geq t\left\{F(kz, hz'; \phi(x)), t\left\{F(fz, kz; \phi(x)), F(gz', hz'; \phi(x))\right\}\right\}$$
$$= t\left\{f(z, z; \phi(x)), t\left\{F(z, z; \phi(x)), F(gz', z; \phi(x))\right\}\right\}$$
$$= F(z, gz'; \phi(x)), \quad \text{implying}$$
$$z = gz'$$

Also, by the weak commutativity of g and h

$F(ghz', hgz'; x) \geq F(hz', gz'; x) = F(z, z; x) = 1,$

giving $gz = ghz' = hgz' = hz$.

Now by (1), we have

$$F(z, gz; x) = F(fz, gz; x)$$
$$\geq t\left\{F(kz, hz; \phi(x)), t\left\{F(fz, kz; \phi(x), F(gz, hz; \phi(x))\right\}\right\}$$
$$= F(z, gz; \phi(x))$$

This gives $z = gz$.

Thus z is the common fixed point of f, g, h and k.

A similar proof works for the case when h is continuous instead of k.

We now suppose that f is continuous. In this case $fku_{2n} \to fz$ and also $f^2u_{2n} \to fz$. By the weak commutativity of f and k, for $\varepsilon > 0$,

$$\begin{aligned} F(kfu_{2n}; fz; \varepsilon) &\geq t\left\{ F(kfu_{2n}, fku_{2n}; \varepsilon/2), F(fku_{2n}, fz; \varepsilon/2) \right\} \\ &\geq t\left\{ F(fu_{2n}, ku_{2n}; \varepsilon/2), F(fku_{2n}, fz; \varepsilon/2) \right\} \\ &> 1 - \lambda . \end{aligned}$$

Therefore $kfu_{2n} \to fz$. Now with the application of (1) we can prove $fz = z$.

As in the previous case, one easlily shows $hz = gz = z$. Since $g:X \to k(X)$, there exists z'' in X such that $gz = z = kz''$. Using (1) again we obtain $fz'' = z$. Since f and k commute weakly, for $x > 0$, we have

$$F(fkz'', kfz''; x) \geq F(kz'', fz''; x) = F(z, z; x) = 1,$$

implying

$$z = fz = fkz'' = kfz'' = kz .$$

Thus z is a common fixed point of f, g, h and k. An analogous proof may be furnished when g is taken continuous in place of f.

The uniqueness of z again follows from (1).

Theorem 2. Let (M, d) be a complete metric space and

$h, k : M \to M, f : M \to h(M), g : M \to k(M)$

satisfy

$$d(fu, gv) \geq \psi(d(ku, hv), d(fu, ku), d(gv, hv))$$

for all $u, v \in M$, where $\psi : R^+ \to R^+$ is an increasing function such that ψ^{-1} is right upper semicontinuous, $\psi(x) < x$ and $\lim\limits_{x} (x - \psi(x)) = \infty$ for all $x \in R^+$. If f and g weakly commute with k and h respectively and one of f, g, h, k, is continuous, then f, g, h, and k have a unique common fixed point. Indeed, the pairs (f, k) and (g, h) have unique common fixed points and these two points coincide.

Proof: We note that if (M, d) is a metric space, then induces a mapping $\mathcal{F}: M \times M \longrightarrow L$, where $\mathcal{F}(v, v) = F(u, v; x) = H(d - d(u, v))$ were H is a distribution function such that $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$. Further, if the t-norm is defined by $t(x, y) = \min\{x, y\}$, then $(M, \mathcal{F}, t)$ is a Menger space. Taking $\psi^{-1} = \phi$, the proof is complete by Theorem 1.

Remark. One may improve Theorems 1-2 by replacing weak commutativity of the maps by their compatibility; and while doing so, the relevant part of the proof (cf. Th.1) is slightly simplified.

REFERENCES

[1] Cheng Chun Chang, On a fixed point theorem of contractive type, *Comm. Math. Univ. St. Paul.*, **32**(1983), 15-19.

[2] Shih-Sen Chang and Kang Shi-Kun, on a generalization of Sehgal-Bharucha-Reid-Istraresch fixed point theorem, *J.Chengdu Univ.Sci. Techno.*, 1 (1983), 103-109.

[3] R.Dedeic and N. Sarapa, on common fixed point theorems for commuting mappings on Menger spaces, *Radovi Mat.* 4(1988), 269- 278.

[4] M.L.Diviccara and S. Sessa, Some remarks on common fixed points of four mappings, *Jñānābha* , **15** (1985), 139-149.

[5] B.Fisher, Common fixed points of four mappings, *Bull.Inst. Math. Acad. Sinica* 8 (1981), 131-140.

[6] B.Fisher, common fixed points of four mappings, *Bull. Inst. Math. Acad. Sinica*, **11** (1983),103-113.

[7] B.Fisher, A common fixed point theorem for four mappings on a compact metric space, *Bull.Inst. Math. Acad. Sinica* 12(1984),249- 252.

[8] T.L. Hicks, Fixed point theory in probabilistic metric spaces, *Zbornik Radova Prirodno-Matematckog faculteta Novi Sad,* **13**(1983), 63-72.

[9] M.Imdad and M.S. Khan, Fixed point theorems for a class of mappings, *Indian J.Pure Appl. Math.*, **14**(1983), 1220-1227.

[10] G.Jungck, Periodic and fixed points and commuting mapping, *Proc. Amer. Math. Soc.*, 76(1979),333-338.

[11] G.Jungck, Compatible Mappings and common fixed points, *Internat. J.Math. and Math. Sci.*, **9**(1986), 771-779.

[12] G.Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.*, **103**(1988), 977-987.

[13] S.N. Mishra, Common fixed points of compatible mappings in *PM*- spaces, *Math.Japon.,* **36**(1991), 283-289.

[14] B.Schweizer,H. Sherwood and R.M. Tardiff, Contractions on probabilistic metric spaces: examples and counter-examples, *Stochastica XII*-**1** (1988), 5-17.

[15] B.Schweizer and A.Sklar, *Probabilistic Metric Spaces*, North Holland Series, 1983.

[16] V.M.Sehgal and A.T.Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, *Math.Sustems Theory,* **6**(1972), 97-102.

[17] Salvatote Sessa,On a weak commutativity condition of mappings in fixed point considerations, *Publ.Inst.Math. (Beograd) (N.S.),* **32** (46) (1982), 149-153.

[18] Salvatore Sessa and B.Fisher, On fixed points of weakly commuting mappings in compact metric spaces, *Jñānābha* 15(1985), 79-91.

[19] Zhang Shi-Sheng (Shih-Sen Chang),Basic theory and applications of probabilistic metric spaces (I), *Appl. Math. Mech. (Eng. Edition),* **9** (1988), No. 2, 123 -133.

[20] S.L. Singh, S.N. Mishra and B.D. Pant, General fixed point theorems in probabilistic metric and uniform spaces, *Indian J. Math.,* **29**(1987), 9-21.

[21] S.L.Singh and B.D.Pant, Coincidence and fixed point theorems for a family of mapping on Menger spaces and extension to uniform spaces, *Math. Japon.,* **33**(1988), 957-973).

[22] S.L.Singh and S.P.Singh, A fixed point theorem, Indian J.Pure Appl. Math. 11 (1980), 1584-1586.

[23] Mila Stojakovic, Common fixed point theorems in complete metric and probabilistic metric space, *Bull. Austral. Math.Soc.*, **36**(1987), 73-88.

[24] Mila Stojakovic, On some classes of contraction mappings, Math.Japon. 33(1988), 311-318.

[25] B.M.L.Tiwari and S.L. Singh, A note on recent generalizations of Jungck contraction principle, *J.UPGC. Acad. Soc.*, **3**(1986), 13-18.

[26] Ding Xieping, Some common fixed point theorems of commuting mappings, *Math.Sem. Notes Kobe Univ.*, **11**(1983), 301-305.

[27] Ding Xieping, A common fixed point theorem of commuting mappings in probabilistic metric space, *Kexue Tongbao*, **29**(1984), 147-150.

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ON CHARACTERIZATIONS OF FIXED POINTS

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ABSTRACT

Equivalent conditions are given for the existence of fixed points of self maps on metric and compact metric spaces.

1. INTRODUCTION

Jungck [1, 2] and Leader [3] employed commuting maps to obtain two criteria for the existence of fixed points of continuous maps on compact metric spaces.

The purpose of this paper is to establish some characterizations of fixed points for self maps on metric and compact metric spaces, some of which are variants of the corresponding results of Jungck [1, 2] and Leader [3].

Let f be a self map on a metric space (X, d), N the set of positive integers. For $x \in X$, define $O(x,f) = \{f^n \mid n \in N \cup \{0\}\}$. $\overline{O(x,f)}$ and $O'(x,f)$ denote the closure and derived of $O(x,f)$ respectively. For $D \subset X$, $\delta(D)$ denotes the diameter of D. f is called compact on D if there exists a compact subset Y of D containing fD. Set

$$C_t = \{g \mid g : X \to X, gf = fg\}$$

$$CX_t = \{g \mid g : X \to X, gf = fg \text{ and } gX = X\}$$

$$A_t = \{g \mid : X \to X, g \cap_{n \in N} f^n X = \cap_{n \in N} f^n X$$

Clearly, the identity map belongs to $CX_t \subset C_t$.

2. LEMMAS

Lemma 2.1. Let f be a continuous self map of a metric space (X, d). Assume $\{A_n\}_{n \in N}$ is a sequence of nonempty compact subsets of X such that $A_{n+1} \subset A_n$ for $n \in N$. Then $f \cap_{n \in N} A_n = \cap_{n \in N} fA_n$ and $f \cap_{n \in N} A_n$ is compact.

Proof. Since X is a metric space and $A_n \supset A_{n+1} \neq \phi$ for $n \in N$, $\cap_{n \in N} A_n$ is a nonempty compact subset by the compactness of A_n, $n \in N$. By the continuity of f it follows that $f \cap_{n \in N} A_n$ is a nonempty compact subset. Clearly $f \cap_{n \in N} A_n \subset \cap_{n \in N} fA_n$. We now prove that $f\cap_{n\in N}A_n \supset \cap_{n\in N} fA_n$. For each $x \in \cap_{n\in N} fA_n$, there exists a sequence $\{a_n\}_{n\in N}$ such that $x = fa_n$ and $a_n \in A_n$ for $n \in N$. For $m \in N$, note that $\{a_n\}_{n\geq m} \subset A_m$ and that A_m is compact, then there exists a convergent subsequence $\{a_{n_k}\}_{k\in N}$ of $\{a_n\}_{n\geq m}$ with linit, say, p. Thus $p \in A_m$ Since A_m is closed. This implies $p \in \cap_{n\in N}A_n$. By the continuity of f, we obtain $x = fa_{n_k} = fp$; i. e. , $x \in f\cap_{n\in N}A_n$. Thus $f\cap_{n\in N}A_n \supset \cap_{n\in N}fA_n$. Hence $f\cap_{n\in N}A_n = \cap_{n\in N}fA_n$. This completes the proof.

Lemma 2.2. Let f be a continuous self metric space (X, d), Y a compact subset of X with $Y \cap fY$. If $A = \cap_{n\in N} f^n Y$, then A is a nonempty compact subset of X and $fA = A$.

Proof. Clearly $f^n Y$ is compact and $f^n Y \supset f^{n+1} Y$ for $n \in N$. By Lemma 2.1 it follows that $fA = \cap_{n\in N} f^{n+1} Y = A$ and that A is a nonempty compact subset of X. This completes the proof.

From Lemma 2.2 we have

Lemma 2.3. Let f be a continuous self map of a compact metric space $(X. d)$. Then $\{f^n \mid n \in N \cup \{0\}\} \subset A_t$.

Lemma 2.4. Let (X, d) be a metric space. Assume there exists a connected subset A of X containing at least two distinct points. Then A is uncountable.

Proof. See [4].

3. FIXED POINTS ON METRIC SPACES

The following result describes an interdependence between connectedness and fixed point.

Theorem 3.1. For a self map f on a metric space (X, d) the following conditions are equivalent.

(1) f has a fixed point;

(2) There exists nonempty countable connected subset A of X satisfying $fA \subset A$;

(3) There exists countable connected subset A of X satisfying $fB \subset B$ for some nonempty subset B of A;

(4) There exists $x \in X$ such that $O(x, f)$ is connected;

(5) Tere exists $x \in X$ such that $\overline{O(x, f)}$ is connected and that $O'(x, f)$ is countable.

Proof. We shall verify the following implications: (1) → (2) → (3) → (1), (1) → (4) → (2) and (1) → (5) → (3). Let (1) holds and x be a fixed point of f. Take $A = O(x, f)$. Then $O'(x, f) = \phi$, $O(x, f) = A = \{x\}$. Hence (2), (4) and (5) holds,. Clearly (2) implies (3).

Let (3) holds. Note that A is countable and connected. By Lemma 2.4 it follows that A is a singleton. Let $A = \{a\}$ for some $a \in X$. Since there exists a nonempty subset B of A such that $fB \subset B$ of $B = A$. This implies $a = fa$. Hence **(1) holds.**

Let (4) holds. Take $A = O\ (x, f)$. Then $fA = O\ (fx, f) \subset A$ and A is countable. Hince (2) holds.

Let (5) holds. Take $A = \overline{O\ (x, f)}$ and $B = O\ (x, f)$. This B and $O'\ (x, f)$ are countable. So is A. Clearly $\phi \neq B \subset A$ and $fB \subset B$. Hence (3) holds, This completes the proof.

For contractive maps we have the folowing two results.

Theorem 3.2. Let f be a continuous self map of a metric space (X, d). Assume there exist $m, n \in N \cup \{0\}$ such that $f^m x \neq f^n y$ implies

$$(*) \qquad d\ (f^m x, f^n y) < \delta\ (\overline{O(x, f)} \cup \overline{O\ (y, f)}\ )$$

Then (1) is equivalent to each of the following:

(6) There exists $x \in X$ such that $O\ (x, f)$ is compact;

(7) There exists $x \in X$ such that $\overline{O\ (x, f)}$ is compact;

(8) There exists $x \in X$ and $p \in N$ such that f^p is compact on $O\ (x, f)$.

Proof. We shall verify the following implications: (1) → (6) → (7) → (1) and (1) → (8) → (6). Clearly (1) implies (6) and (8).

Let (6) hold. Since $O\ (x, f)$ is compact and X is a metric space. $O\ (x, f)$ is closed. Thus $\overline{O\ (x, f)}$ is compact. Hence (7) holds.

Let (7) hold. Since f is continuous, $f\,\overline{O(x, f)} \subset \overline{fO(x, f)} \subset \overline{O(x, f)}$. Set $A = \cap_{n \in N} f^n\ \overline{O(x, f)}$. By Lemma 2.2 it follows that A is a nonempty compact subset of X and $fA = A$. We claim that A is a singleton. If not, then $\delta\ (A) > 0$. By the compactness of A there exist $u, v \in A$ such that $d\ (u, v) = \delta\ (A)$. Obviously we can find $a, b \in A$ such that $f^m a = u, f^n b = v$. Note that $\overline{O\ (a, f)} \cup \overline{O\ (b, f)} \subset \overline{A} = A$. Using (*)

$$0 < d\ (f^m a, f^n b) < \delta\ (\overline{O\ (a, f)} \cup \overline{O\ (b, f)}) \leq \delta\ (A)$$

which implies

$$\delta\ (A) = d\ (u, v) = d\ (f^m a, f^n b) < \delta\ (A)$$

which is impossible. Thus A is a singleton. Let $A = \{w\}$. Then $Fw = w$. Hence (1) holds.

Let (8) holds. Since f^p is compact on $O\ (x, f)$, there exists a compact subset Y of $O'\ (x, f)$ containing $f^p\, O\ (x, f)$. Note that $O\ (x, f) - Y \subset O\ (x, f) - f^p\ O\ (x, f) = \{x, fx, \ldots, f^{p-1} x\}$. Then $O\ (x, f) - Y$ is finite. This implies $O\ (x, f)$ - Y is a compact subset of $O\ (x, f)$. Clearly $O\ (x, f) = Y \cup (x, f) - Y)$ and $O\ (x, f)$ is compact. Hence (6) holds. This completes the proof.

Theorem 3.3 Let f be a continuous self map of a metric space

(X, d) satisfuing

$$d(fx, fy) < \delta\, \overline{(O(x, f)} \cup \overline{O(y, f))}$$

for distinct $x, y \in X$.. Then (1), (6), (7) and (8) are equivalent.

Since the proof of Theorem 3.3 is similar to that of Theorem 3.2, so we omit the proof.

4. FIXED POINTS ON COMPACT METRIC SPACES

In this section we give eight necessary and sufficient conditions for the existence of fixed points of a continuous map on compact metric spaces.

Theorem 4.1. Let f be a continuous self map of a compact metric space (X, d). Then (1) is equivalent to each of the following:

(9) There exists $x \in X$ such that $\lim\limits_{n \to \infty} d(f^n x, f^{n+1} x) = 0$;

(10) there exists $x \in X$ such that $\lim\limits_{n \to \infty} d(f^n x, f^{n+1} x) = 0$;

(11) For each $g \in CX_t$, $gx \neq gy$ implies

$(*)'$ $\qquad d(gx, gy) > inf\{d(hx, hy) \mid h \in C_t\}$

(12) There exists $g \in CX_t$ such that $gx \neq gy$ implies $(*)'$ holds;

(13) For each $g \in A_t$, $gx \neq gy$ implies $(*)'$ holds

(14) There exists $g \in A_t$ such that $gx \neq gy$ implies $(*)'$ holds.

Proof. We shall verify the following implications : (1) → (9) → (10) → (1) → (11) → (12) → (1) and (1) → (13) → (14) → (1). Clearly (1) implies (9), (9) implies (10), (11) implies (12) and (13) implies (14).

Let (10) holds. Since $\lim\limits_{n \to \infty} inf\, d(f^n x, f^{n+1}, x) = 0$, there exists a strictly increasing sequence $\{n_i\}_{i \in N} \subset N$ such that $\lim\limits_{i \to \infty} d(f^{n_i} x, f^{n_i + 1} x) = 0$. Note that X is a compact metric space. Without loss of generality we assume that $\lim\limits_{i \to \infty} f^{i_n} x = a \in X$. By the continuity of f and d we have $d(a, fa) = \lim\limits_{i \to \infty} d(f^{n_i} x, f^{n_i + 1} x) = 0$; i. e. $a = fa$. Hence (1) holds.

Let (1) holds and w be a fixed point of f. Define t:- $X \to X$ by $tx \equiv w$ for $x \in X$. Then $t \in C_t$ and $d(tx, ty) \equiv 0$ for $x, y \in X$. if $g \in CX_t \cup A_t$ and $gx \neq gy$, then $d(gx, gy) > 0 = inf\{d(hx, hy) \mid h \in C_t\}$. Hence (11) and (13) hold.

Let (12) holds. By the continuity of f and compactness of X, there exists $a \in X$ such that $d(a, fa) = inf\{d(x, fx) \mid x \in X\}$. It follows from $g \in CX_t$ that there exists $x \in X$ such that $gx = a$. We assert that $a = fa$. Otherwise $a \neq fa$; i. e. , $gx \neq fgx = gfx$. Using (*)′

$$d(a, fa) = d(gx, gfx) > inf\{d(hx, hfx) \mid h \in C_t\}$$

$$= inf\{d(hx, fhx) \mid h \in C_t\} \geq d(a, fa)$$ which is a contradiction. Hence $a = fa$. Thus (1) holds.

Let (14) holds Set $A = \cap_{n \in N} f^n X$. By Lemma 2.2 it follows that A is a nonempty compact subset of X and $fA = A$. Since f is continuous and A is compact, there exists $a \in A$ such that $d(a, fa) = inf\{d(x, fx) \mid x \in A\}$. Note that $g \in A_t$. Then there exists $x \in A$ such that $gx = a$. Similarly we can prove that $a = fa$. Hence (1) holds. This completes the proof.

Note that the identity map $\in CX_t$ and that $f \in A_t$ by Lemma 2.3. Then the following result follows fuom Theorem 4.1.

Corollary 4.2. Let f be a continuous self map of a compact metric space (X, d). Then (1) is equivalent to each of the following:

(15) $x \neq y$ implies $d(x, y) > inf\{d(hx, hy) \mid h \in C_t\}$;

(16) $fx \neq fy$ impies $d(fx, fy) > inf\{d(hx, hy) \mid h \in C_t\}$.

Remark. Corollary 4.2 is variants of the corresponding results of Jungck [1,2] and Leader [3].

REFERENCES

[1] S.Leader, Uniformly contractive fixed points in compact metric spaces, *Proc. Amer. Math. Soc.* **86** (1982), 153-158.

[2] G.Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly* **83** (1976), 261-263.

[3] G.Jungck, Periodic and fixed points and commuting mappings, *Proc. Amer. Math. Soc.* **76** (1979), 333-338.

[4] C.Berge, *Topological spaces*, Oliver & Boyd, London, 1963,, 96-67.

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ALMOST PERIODIC FUNCTIONS ON FUZZY METRIC SPACE

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1. The notion of fuzzy set is introduced by Zadeh *[7]*. Here all the properties of an interval is not required. Let $M : X \to I := [0, 1]$. $M(x)$ the degree of membership of a point x in X is a fuzzy set M. It is a generalization of characteristic function. General topology has been applied systematically first by Chang [1]. There are mainly two ways of defining fuzzy metric. In one the numerical distances are measured between fuzzy objects. This type was studied by Erceg [3], Deng [2] etc. Second approach deals with the distance between objects called fuzzy. The objects are also fuzzy. We refer to Kuleva-Seikkalu [5], Eklund and Gahler [4].

We need some definitions due to Sharma, Singh and Mishra [6]:

Definition 1. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous $\in$ - norm if $([0, 1], *)$ is an abelian (topological) monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ *and* $b \leq d$ *(a, b, c, d are in [0, 1])*.

Definition 2. The 3-tuple $(X, M, *)$ is a fuzzy metric space if X be arbitrary set, $*$ is continuous $\in$ - norm, and M is fuzzy set $X^2 \times (0, \infty)$ satisfying :

(1.1) $m(x, y, 0) = 0$,

(1.2) $M(x, y, \in) = 1$ for all $t > 0$ iff $x = y$.

(1.3) $M(x, y, \in) = M(y, x, \in)$,

(1.4) $M(x, y, \in_1)^* M(y, z, \in_2) \leq M(x, z, \in_1 + \in_2)$

(1.5) $M(x, y, .) : (0, \infty) \to [0, 1]$

is left continous for all x, y, z in X and $C_1, C_2 > 0$.

Now for the first time we define almost periodic functions with values in fuzzy metric space.

Difinition 3. A continuous function on R (real). $T : R \to (X, M, *)$ is called almost periodic if for any member $\in > 0$, one can find $l(\in) > 0$ such that any interval of length $l(\in)$ contains at least one point $\tau_{\in}$ with property:

$$M\left(T(t+\tau), T(t), \in\right) = 1, t \in R$$

2. Now we shall prove:

Theorem 1. An almost periodic function on fuzzy metric space is uniformly contionuous.

Proof. Let $l = l(\in/3) > 0$ which corresponds to $\in/3, \in > 0$ in the definition of almost periodic function. On the interval $[-1, 1+l]$ the function is continuous and hence it is also uniformly continuous. Let $\delta = \delta(\in/3) > 0, \delta < 1$ such that $t_1, t_2 \in [-1, 1+l]$ with $|t_1 - t_2| < \delta$, we have

$$M(Tt_1, Tt_2, \tfrac{\epsilon}{3}) = 1 .$$

Finally, let $t_1 > t_2$ be such that $|t_1 - t_2| < \delta$ and τ be any $(\in/3)$ translation number of T contiained in the interval $[-t_1, -t_1 + 1]$. Since $|t_2 - t_1| < \delta, 0 \le t_1 + \tau \le l$ it follows that $t_2 + \tau$ is situated on the interval $[-1, 1+l]$. Thus

$$\begin{aligned} M(Tt_2, Tt_1, \in) &\ge M\ [Tt_2, T(t_2+\tau), \in/3]^* \\ & M\ [T(t_2+\tau), T(t_1+\tau, \in/3\ ] \\ & *\ M\ [(Tt_1+\tau), T(t_1), \in/3] \\ & = 1 * 1 * 1 = 1 \end{aligned}$$

This proves the result.

Theorem 2. If T_n is a sequence of almost periodic function with values in *(X, M, *)* and if

$$\lim_{n \longrightarrow \infty} T_n(t) = T(t)$$

Uniformly in the sense of convergence of $\in$ —norm, then T is almost periodic.

Proof. Continuity of T is evident. If $\in > 0$, then there exists a natural number $N(\in)$ with

(2.1) $M(T_n t, Tt, \in/3) = 1,\ t \in R,\ n \ge N(\in)$.

We fix n_o, for which (2.1) is true and consider $l(\in/3)$ determined from the almost periodicity of T_{n_o} and T an $(\in/3)$ translation number of f_{no}. For any $t \in R$, We have

$$\begin{aligned} & M(T(t+\tau), T(t), \in) \ge M(T(t+\tau), (T_{n_o}(t+\tau), \in/3) \\ & *\ M(T_n(t+\tau), T_n t, (\in/3) * M(T_n t, Tt, \in/3) \\ & = 1 * 1 * 1 = 1 \end{aligned}$$

which proves the almost periodicity of T.

REFERENCES

[1] Chang C., Fuzzy topological sapce, *J.Math. Anal. Appl.*, **24** (1968), 182-190.

[2] Deng, Z., Fuzzy pseudo-metric spaces, *J. Math. Anal. Appl.*, **86** (1982), 74-95.

[3] Erceg, M.A., Metric spaces in fuzzy set theory, *J.Math. Anal. Appl.*, **69** (1979), 205-230.

[4] Eklund, P. and Gahler, W. Basic notions for fuzzy topology, *Fuzzy Sets and Systems,* **26** (1988), 333-356.

[5] Kaleva, O. and Seikkala, S. On fuzzy metric spaces, *Fuzzy Sets and Systems,* **12** (1984), 215-229.

[6] Nilima Sharma, Singh, S.L. and Mishra, S.N., Contraction mapping in fuzzy metric spaces, *International Journal of Mathematics* (U.S.A.) (In press).

[7] Zadeh, L.A. Fuzzy sets as a basis for a theory possibility. *Fuzzy Sets and Systems,* **1** (1978), 3-29.

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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

WHEN IS THE WALLMAN COMPACTIFICATION FINITE ?

By

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ABSTRACT

In [1], the following questions were posed as some of the open problems:

(1) Suppose that X is a T_1 space. What are the necessary and sufficient conditons that the Wallman compactification be a one-point compactification?

(2) More generally, what are the necessary and sufficient conditions that the Wallman compactification be a finite compactification with n points?

The purpose of this note is to present solutions to these problems.

Background notation and results.

Let X be a set, and $\mathcal{B}$ a filter-base on X. $\mathcal{B}$ is said to be free if the intersection of all members of $\mathcal{B}$ is empty.

If X is a topological space, a filter-base $\mathcal{B}$ is called a *closed base* if each member of $\mathcal{B}$ is closed $\mathcal{B}$ is called. a *maximal closed base* if it is a closed base and is not properly contained in any closed base. Maximal closed bases were used by Wallman [2] to construct a T_1 compactification of an arbitrary T_1 space. It is well-known that the Wallman compactification of a Tychonoff space X coincides with the Stone-Cech compactification βX iff X is a T_4 space. Necessary and sufficient conditions are also known for βX to be of finite cardinality n, with the case $n = 1$ attracting the attention of a large number of researchers. This prompted the questions posed at the beginning of this article. At first sight, it may appear that the solution to our problem could be obtained by simply combining the conditions for coincidence of Wallman and Stone-Cech compactifications with the conditions for coincidence of Stone- Cech and Alexandroff compactifications. However, in our problem we require X to satisfy the T_1 condition only. Hence the Stone- Cech compactification need not exist. But the Wallman and Alexandroff compactifications certainly exist and are T_1 spaces. The solutions we present are surprisingly simple-looking conditions. In proving our theorem, we will use the following well-known properties of the Wallman compactification.

In what follows, X shall denote an arbitrary noncompact T_1

space, and W its Wallman compactification.

1. THEOREM. *[2]*

(1) The growth $W - X$ consists of all free maximal closed bases.

(2) If A and B are disjoint, closed subsets of X, the their W- closures are also disjoint.

Main result.

2. THEOREM. In order for $[W - X] = 1$, it is necessary and sufficient that X satisfy the condition:

Of every pair of disjoint closed sets, at least one is compact.

Proof: Necessity: Suppose that $W - X$ is the singleton $\{\infty\}$. Let A and B be closed, disjoint subsets of X. By Theorem 1, $\overline{A} \cap \overline{B} = 0$, where the bar designates closure in W. Hence, ∞ cannot belong to both $\overline{A}$ and $\overline{B}$. Say, $\infty \notin \overline{A}$. Thus, $\overline{A}$ is closed and compact in W, hence also closed and compact in X. Further, the X-closure of A coincides with its W-closure. This proves that A is compact in X.

Suffciency: Suppose that $W–X$ is not a singleton. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two distinct points of $W–X$ then $\mathcal{B}_1$ and $\mathcal{B}_2$ are free maximal closed bases. Because of their maximality, there exist $B_i \in \mathcal{B}_i$ $(i = 1, 2)$ such that $B_1 \cap B_2 = 0$ If one of them, say B_1, were compact, then the open cover $\{X - A \mid A \in \mathcal{B}\}$ must contain a finite subcover of B_1:

$$X - A_1, X - A_2, \dots, X - A_n .$$

This implies that $A_1 \cap A_2 \cap \dots \cap A_n \cap B_1 = 0$, which contradicts $\mathcal{B}_1$, being a filter-base. Therefore B_1 and B_2 are not compact.

The above theorem, as well as its proof, is easily generalized to the case of n points.

3. THEOREM. The following are equivalent:

(1) $| W - X | \leq n$.

(2) If $A_1, A_2, \dots, A_{n+1}$ are pairwise disjoint, closed subsets of X, then at least one A_i is compact.

REFERENCES

[1] Murdeshwar, Mangesh: A characterization of Alexandroff Uniformity, *Q and A in Gen. Top.* **9** (1991), 145-150.

[2] Wallman, Henry: Lattices and topological spaces, *Ann. Math.*, **39** (1938), 112-126.

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A DIFFERENCE FORMULA AND SOME LINEAR GENERATING RELATIONS FOR THE DISCRETE VARIABLE ANALOGUE OF MODIFIED JACOBI POLYNOMIALS

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ABSTRACT

In the present paper, we have defined a complex variable discrete analogue of Modified Jacobi polynomials, viz. $S^{\alpha}_{m+n}(z, \lambda, \mu, w)$ by using the theory of finite differences, by means of a $(m+n)$th partial difference formula. In this paper we have established the difference formula, addition and multiplication theorems, linear generating relations and some other interesting results.

1. Definition.

We define our polynomial set $S^{\alpha}_{m+n}(z, \lambda, \mu, w)$ by means of a $(m+n)$th partial difference formula :

$$S^{\alpha}_{m+n}(z, \lambda, \mu, w) = \frac{(1+\mu w)^{z/w}}{(m+n)\,!(z-\lambda w)^{[\alpha w]}} \underset{w}{\overset{n}{\Delta}} x \underset{w}{\overset{m}{\Delta}} iy[(z-\lambda w)^{[\overline{\alpha+m+n}\,w]} \cdot (1+\mu w)^{-z/w}], \qquad \ldots (1.1)$$

wherein $\underset{w}{\Delta} x$ and $\underset{w}{\Delta} iy$ are partial difference quotients, which is valid for discrete values of the variables at equal intervals viz. $(x, x-w, \ldots, x-nw)$ and $iy, (iy-w, \ldots, iy-mw)$. As $w \to 0$, the variables become continuous. In the limiting case ($w \to 0$ with $\mu = 1$) our polynomial set takes the form of the extended Laguere polynomial $L^{(\alpha)}_{N}(z)$, where $z = (x+iy)$, α is a non-negative integer. μ is a positive real numbber and λ is any complex number say $\lambda_1 + i\lambda_2$ such that $Re(\lambda w)$ is a natural number. We shall be using the following formulae:

$$\text{(i)}\quad \underset{w}{\Delta x} f(x) = \frac{1}{w}[f(x+w) - f(x)] \Rightarrow \lim_{w \to 0} \underset{w}{\Delta x} f(x) = \frac{d}{dx} f(x) \qquad \ldots (1.2)$$

$$\text{(ii)}\quad x^{[\alpha w]} = x(x-w)(x-2w) \ldots (x-\alpha w+w)\ , \qquad \ldots (1.3)$$

$$\text{(iii)}\ \overset{n}{\underset{w}{\Delta}}x\,(u_x\, v_x) = \sum_{r=0}^{n} {}^nC_r \overset{n-r}{\underset{w}{\Delta}}(u_{x+rw})\, \overset{r}{\underset{w}{\Delta}}x\,(v_x) \qquad \text{... (1.4)}$$

$$\text{(iv)}\ \overset{r}{\underset{w}{\Delta}}x\, x^{[\alpha w]} = \frac{r\,!}{(\alpha - r)\,!}\, x^{[\overline{\alpha - r}w]} \qquad \text{... (1.5)}$$

$$\text{(v)}\ \overset{r}{\underset{w}{\Delta}}x\,(1+bw)^{-x/w} = (-b)^r\,(1+bw)^{-\frac{x}{w}-r} \qquad \text{... (1.6)}$$

$$\text{(vi)}\ (x-\lambda w + rw)^{[\overline{\alpha+r}]} = w^r\left(\frac{x}{w}-\lambda+1\right)_r (x-\lambda w)^{[\alpha w]} \qquad \text{... (1.7)}$$

$$\text{(vii)}\ \overset{n}{\underset{w}{\Delta}}x\left(\frac{x}{w}\right)_R = \frac{(R-n+1)}{w^n}\left(-\frac{x}{w}+n\right)_{R-n} \qquad \text{... (1.8)}$$

The explicit form of our polynomial set comes out to be

$$S^{\alpha}_{m+n}(z,\lambda,\mu,w) = \frac{(1+\alpha)_{(m+n}}{(m+n)\,!}\, F_1\left[\frac{z}{w}-\lambda+1,-m,-n;1+\alpha;\frac{\mu w}{1+\mu w}\right] \qquad \text{... (1.9)}$$

$$= \frac{(1+\alpha)_{m+n}}{(m+n)\,!}\, 2F_1\left[\frac{z}{w}-\lambda+1,-m-n;1+\alpha;\frac{\mu w}{1+\mu w}\right] \qquad \text{... (1.10)}$$

$$= P_{m+n}^{(\alpha,\,\frac{z}{w}-\lambda-\alpha-m-n)}\left(\frac{1-\mu w}{1+\mu w}\right), \qquad (1.11)$$

where $P_N^{\alpha_1,\,\beta_1-N}(w)$ are the modified Jacobi polynomials.

As $w \to 0$ and $\mu = 1$, we have

$$\lim_{w\to 0} S^{\alpha}_{m+n}(z,\lambda,l,w) = \frac{(1+\alpha)_{m+n}}{(m+n)\,!}\, 1F_1\,[-m-n;1+\alpha;z] \qquad \text{... (1.12)}$$

$$= L^{(\alpha)}_{m+n}(z), \qquad \text{... (1.13)}$$

where we name $L^{(\alpha)}_N(z) \equiv L^{(\alpha)}_{m+n}(z)$ as the extended Laguerre polynomials.

If, however, $m=0$ and simultaneously, the imaginary part of z vanishes, we arrive at the set of classical Laguerre polynomials $L^{(\alpha)}_n(x)$.

2. Difference Formula :

$$\overset{p}{\underset{w}{\Delta}}x\,\overset{q}{\underset{w}{\Delta}}iy\, S^{\alpha}_{m+n}(z,\lambda,\mu,w)$$

$$= \frac{(-\mu)^{p+q}}{(1+\mu w)^{p+q}}\, S^{\alpha+p+q}_{m+n-p-q}(z+p+q,\lambda,\mu,w) \qquad \text{... (2.1)}$$

Proof. We have the left hand side of (2.1)

$$= \frac{(1+\alpha)_{m+n}}{(m+n)\,!} \underset{w}{\overset{p}{\Delta}} x \, \underset{w}{\overset{q}{\Delta}} iy \sum_{k=0}^{m+n} \frac{(-m-n)_k}{k\,!\,(1\ \alpha)_k} \left(\frac{x+iy}{w}-\lambda+1\right)_k \left(\frac{\mu w}{1+\mu w}\right)^k$$

$$= \frac{(1+\alpha)_{m+n}}{(m+n)\,!} \sum_{k=p+q}^{m+n} \frac{(-m-n)_k \left(\frac{x+iy}{w}-\lambda+1+p+q\right)_{k-p-q} \left(\frac{\mu w}{1+\mu w}\right)^k}{(k-p-q)\,!\,(1+\alpha)_k\, w^{p+q}}$$

$$= \frac{(1+\alpha)_{m+n}\,(-m-n)^{p+q}\,\mu^{p+q}}{(m+n)\,!\,(1+\alpha)^{p+q}\,(1+\mu w)^{p+q}}$$

$$\sum_{k=0}^{m+n-p-q} . \frac{(-m-n+p+q)_k \left(\frac{x+iy}{w}-\lambda+1+p+q\right)_k \left(\frac{\mu w}{1+\mu w}\right)^k}{k\,!\,(1+\alpha+p+q)_k}$$

$$= \frac{(1+\alpha+p+q)_{m+n-p-q}}{(m+n-p-q)\,!} \left(\frac{-\mu}{1+\mu w}\right)^{p+q} {}_2F_1[-m-n+p+q,\ \frac{z}{w}-\lambda+1+p+q;\ 1+\alpha+p+q;\ \frac{\mu w}{1+\mu w}]$$

Hence the result.

3. Addition and Multiplication Theorems.

An appeal to the finite difference analogue of the Taylor's theorems, *viz.*

$$f(x+y) = \sum_{p=0}^{\infty} \frac{y^{[pw]} \left\{\underset{w}{\overset{p}{\Delta}} x\, f(x)\right\}}{p\,!} \qquad \dots (3.1)$$

and $$f(xz) = \sum_{p=0}^{\infty} \frac{\{x(z-1)\}^{[\mathrm{pw}]} \left\{\underset{w}{\overset{p}{\Delta}} x\, f(x)\right\}}{p\,!}, \qquad \dots (3.2)$$

the latter being obtained by putting $y = x(z-1)$ in (3.1), we arrive at the following addition and multiplication theorems for our polynomial set with $m = 0$ and simultaneously making the imaginary part of z vanish :

$$S_n^{\alpha}(x+x_1, \lambda, \mu, w) = \sum_{p=0}^{\infty} \frac{x_1^{[pw]} \left(\frac{-\mu}{1+\mu w}\right)^p S_{n-p}^{\alpha+p}(x+p, \lambda, \mu, w)}{p\,!} \qquad \dots (3.3)$$

$$S_n^{\alpha}(xx_2, \lambda, \mu, w) = \sum_{p=0}^{\infty} \frac{\{x(x_2-1)\}^{[\mathrm{pw}]} S_{n-p}^{\alpha+p}(x+p, \lambda, \mu, w)}{p\,!} \qquad \dots (3.4)$$

4. Linear generating relations .

By using the usual series manipulation method, the following five linear generating relations can be established :

(a) $$\sum_{N=0}^{\infty} \frac{S_N^{\alpha}(z, \lambda, \mu, w)\, [(\gamma_p)]_N\, t^N}{[(\delta_q)]_N}$$

$$= F_{q\ \ :0,1}^{p+1:0,1}\left[\begin{matrix} [(\gamma_p); 1, 1], [1+\alpha; 1, 1]:- ; [\frac{z}{w} - \lambda + 1; 1] \\ [(\delta_q); 1, 1]:- ; [1+\alpha; 1] \end{matrix}\ t, \frac{-\mu w t}{1+\mu w}\right] \qquad \ldots (4.1)$$

(b) $$\sum_{m,n=0}^{\infty} \frac{S_{m+n}^{\alpha}(z, \lambda, \mu, w)(m+n)!\, t^m\, T^n}{(1+\alpha)_{m+n}}$$

$$= (1-t)^{-1}(1-T)^{-1} F_1[\frac{z}{w} - \lambda + 1; 1, 1; +\alpha\, ; \frac{-\mu w t}{(1+\mu w)(1-t)}, \frac{-\mu w T}{(1+\mu w)(1-T)}] \qquad \ldots (4.2)$$

(c) $$\sum_{m,n=0}^{\infty} \frac{S_{m+n}^{\alpha}(z, \lambda, \mu, w) t^m\, T^n}{m\,!\, n\,!\, (1+\alpha)_{m+n}}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\frac{z}{2} - \lambda + 1)_{r+s} \left(\frac{-\mu w}{1+\mu w}\right)^{r+s} t^r\, T^s}{r\,!\, s\,!\, (1)_{r+s}\, (1+\alpha)_{r+s}}\, F_1(- ; , - ; 1 + r + s; t; T] \qquad \ldots (4.3)$$

(d) $$\sum_{m,n=0}^{\infty} \frac{(m+n)\,!\, S_{m+n}^{\alpha}(z, \lambda, \mu, w)}{m\,!\, n\,!} \left(\frac{t}{1+\alpha}\right)^{m+n}$$

$$= (1 - \frac{2t}{1+\alpha})^{\frac{z}{w} - \lambda - \alpha} \left[1 - \frac{2t}{(1+\mu w)(1+\alpha)}\right]^{-\left(\frac{z}{w} - \lambda + 1\right)} \qquad \ldots (4.4)$$

(e) Using the auxiliary variable v, we have

$$\sum_{N=0}^{\infty} \frac{(N+k)\,!\, S_{N+k}^{\alpha}(z, \lambda, \mu, w) T^N}{N\,!\, k\,!}$$

$$= (1-T)^{(-1-\alpha-k)} (1+\mu w)^{\frac{z}{w} - \lambda + 1} (1 + \frac{\mu w}{1-T})^{-(z/w - \lambda + 1)}\, S_k^{\alpha}\left(\frac{z}{1-T}, \lambda, \mu, \frac{w}{1-T}\right) \qquad \ldots (4.5)$$

The following interesting special cases of these linear generating relations may be listed as follows :

$$\text{(i)} \sum_{N=0}^{\infty} S_N^{\alpha}(z, \lambda, \mu, w)\, t^N = (1-t)^{-1-\alpha} \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z}{w} - \lambda + 1\right)} \qquad \dots (4.6)$$

As $w \to 0$ and $\mu = l$, $m = 0$ and simultaneously imaginary part in z vanishes, (4.6) converts into a known linear generating relation for the Laguerre polynomials.

$$\text{(ii)} \sum_{m,\,n=0}^{\infty} \frac{(m+n)!\, L_{m+n}^{(\alpha)}(z)\, t^m T^n}{(1+\alpha)_{m+n}}$$

$$= (1-t)^{-1} (1-T)^{-1}\, \phi_2 \left[1, 1; 1+\alpha; \frac{-zt}{1-t}, \frac{-zt}{1-T}\right] \qquad \dots (4.7)$$

$$\text{(iii)} \sum_{m,\,n=0}^{\infty} \frac{L_{m+n}^{(\alpha)}(z)\, t^m T^n}{m!\, n!\, (1+\alpha)_{m+n}} = \sum_{r,\,s=0}^{\infty} \frac{(-zt)^r (-zT)^s}{r!\, s!\, (1)_{r+s} (1+\alpha)_{r+s}}$$

$$. F_1 [- ; -, -, 1+r+s; t, T] \qquad \dots (4.8)$$

$$\text{(iv)} \sum_{m,\,n=0}^{\infty} \frac{(m+n)!\, L_{m+n}^{(\alpha)}(z)}{m!\, n!} \left(\frac{t}{1+\alpha}\right)^{m+n}$$

$$= \left(1 - \frac{2t}{1+\alpha}\right)^{-1-\alpha} e^{-(2zt/(1+\alpha-2t))} \qquad \dots (4.9)$$

$$\text{(v)} \sum_{N=0}^{\infty} \frac{(N+k)!\, L_{N+k}^{(\alpha)}(z)\, T^N}{N!\, k!}$$

$$= (1-T)^{-1-\alpha-k} \exp(-zt/(1-T))\, L_k^{(\alpha)}(z/(1-T)) \qquad \dots (4.10)$$

5. Some other interesting results :

If $N = m + n$ and $M = m_1 + n_1$ and we use the series manipulation method, we arrive at the following result :

$$\text{(a)} \sum_{N=0}^{\infty} \frac{S_N^{\alpha}(z, \lambda, \mu, w) \left(\frac{t}{2}\right)_N [(\gamma_p)]_N\, w^N}{[(\delta_q)]_N}$$

$$= F_{q\ \ :0;\,1}^{p+2:0;\,1} \left[\begin{matrix} [(\gamma_p); 1, 1], [t/w; 1, 1], [1+\alpha; 1, 1] :- ; \\ [(\delta_q); 1, 1] \qquad :- ; \end{matrix} \quad \begin{matrix} [\frac{z}{w} - \lambda + 1; 1] \\ [1+\alpha; 1] \end{matrix}\ w, \frac{-\mu w^2}{1+\mu w} \right] \qquad \dots (5.1)$$

The rest of the following three results, *viz.*

$$\text{(b) } S_N^{\alpha}(z, \lambda, \mu, w) = \sum_{k=0}^{N} (\alpha - \beta)_k \, S_{N-k}^{\beta}(z, \lambda, \mu, w) \qquad \dots (5.2)$$

$$\text{(c) } S_N^{\alpha+\beta+1}(z_1 + z_2, \lambda_1 + \lambda_2 - 1, \mu, w)$$

$$= \sum_{k=0}^{N} S_{N-k}^{\alpha}(z_1, \lambda_1, \mu, w) \, S_k^{\beta}(z_2, \lambda_2, \mu, w) \qquad \dots (5.3)$$

$$\text{(d) } [\underset{w}{\Delta} x_1 \underset{w}{\Delta} iy_1 S_M^{\alpha}(z_1, \lambda_1, \mu, w)][\underset{w}{\Delta} x_2 \underset{w}{\Delta} iy_2 S_N^{\beta}(z_2, \lambda_2, \mu, w)]$$

$$= \frac{\mu^4}{(1+\mu)^4} \sum_{k=0}^{N-2} \frac{(4)_k (1 + \mu w)^{-k}}{k\,!} S_{M-2}^{\alpha}(z_1, \lambda_1, \mu, w)$$

$$S_{N-k-2}^{\beta}(z_2, \lambda_2, \mu, w), \; M \geq 2, \; N \geq 2 \qquad \dots (5.4)$$

have, however been established below in brief.

(b) From equation (4.6), we have

$$\sum_{N=0}^{\infty} S_N^{\alpha}(z, \lambda, \mu, w) t^N = (1-t)^{\beta-\alpha} \left[(1-t)^{-1-\beta} \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{z/w - \lambda + 1}\right]$$

$$= (1-t)^{\beta-\alpha} \sum_{N=0}^{\infty} S_N^{\beta}(z, \lambda, \mu, w) t^N$$

$$= \sum_{N=0}^{\infty} \sum_{k=0}^{N} \frac{(\alpha-\beta)_k \, S_{N-k}^{\beta}(z, \lambda, \mu, w) t^N}{k\,!}$$

$\Rightarrow$(5.2)

(c) Using (4.6) to the following identity

$$(1-t)^{-2-\alpha-\beta} \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z_1+z_2}{2} - \overline{\lambda_1 + \lambda_2 - 1} + 1\right)}$$

$$= (1-t)^{-1-\beta} \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z_2}{w} - \lambda_2 + 1\right)} (1-t)^{-1-\alpha}$$

$$\cdot \left(1 + \frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(\frac{z_1}{w} - \lambda_1 + 1\right)},$$

we get

$$\sum_{N=0}^{\infty} S_N^{\alpha+\beta+1}(z_1 + z_2, \lambda_1 + \lambda_2 - 1, \mu, w) t^N$$

$$= \sum_{k=0}^{\infty} S_k^{\beta}(z_2 , \lambda_2, \mu, w)t^k \sum_{N=0}^{\infty} S_N^{\alpha}(z_1, \lambda_1, \mu, w)t^N$$

$$= \sum_{N=0}^{\infty} \sum_{k=0}^{N} S_{N-k}^{\alpha}(z_1, \lambda_1, \mu, w)\, S_k^{\beta}(z_2, \lambda_2, \mu, w)t^N,$$

$\Rightarrow$(5.3)

(d) Using (4.6) and

$$\sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r=0}^{m_1k_1+\ldots+m_rk_r<n} B(k_1, \ldots, k_r; n+m_1k_1+\ldots+m_rk_r),$$

$$= \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r=0}^{\infty} B(k_1, \ldots k_r; n+m_1k_1+\ldots m_rk_r),$$

where $m_1, m_2 \ldots, m_r$ are positive integers ($r \geq 1$), we have

$$\sum_{M,N=0}^{\infty} \underset{w}{\Delta} x_1 \underset{w}{\Delta} iy_1 S_M^{\alpha}(z_1, \lambda_1, \mu, w)]\,[\underset{w}{\Delta} x_2 \underset{w}{\Delta} iy_2 S_N^{\beta}(z_2, \lambda_{,} \mu, w)] \;.\; t^{M+N}$$

$$= (1-t)^{-2-\alpha-\beta}\left\{\underset{w}{\Delta} x_1 \underset{w}{\Delta} iy_1 \left(1+\frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(z_1/w-\lambda_1+1\right)}\right\}$$

$$\left\{\underset{w}{\Delta} x_2 \underset{w}{\Delta} iy_2 \left(1+\frac{\mu w t}{(1+\mu w)(1-t)}\right)^{-\left(z_2/w-\lambda_2+1\right)}\right\}$$

$$= \frac{\mu^4 t^4}{(1+\mu w-t)^4}\left\{1+\frac{\mu w t}{(1+\mu w)(1-t)}\right\}^{-\left(\frac{z_1+z_2}{w}-\lambda_1-\lambda_2+2\right)}$$

$$= \frac{\mu^4}{(1+\mu w)^4} \sum_{M, N, k=0} \frac{(4)_k\,(1+\mu w)^{-k}}{k\,!} S_M^{\alpha}(z_1, \lambda_1, \mu, w)\, S_N^{B}(z_2, \lambda_2, \mu, w) \;.\; t^{M+N+k+4}$$

$$= \frac{\mu^4}{(1+\mu w)^4} \sum_{M=2}^{\infty} \sum_{N=2}^{\infty} \sum_{k=0}^{N-2} \frac{(4)_k\,(1+\mu w)^{-k}}{k\,!} S_{M-2}^{\alpha}(z_1, \lambda_1, \mu, w) \;.\; S_{N-k-2}^{\beta}(z_2, \lambda_2, \mu, w)\, t^{M+N},$$

$\Rightarrow$ (5.4)

In (5.4), roles of M and N may however be interchanged.

REFERENCES

[1] Jordan, Charles, *Calculus of finite Differences,* Chelsea Publ. Co., New York, 1950.

[2] Milne Thomson, *The Calculus of finite Differences,* London, 1933.

[3] Rainville, E.D., *Special Functions*L,- Macmillan, New York, 1960, Reprinted by Chelsea Publ. Co. Ltd., Bronx, New York, 1971.

Jñānābha Vol. 23, 1993

(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

THE QUATTRO SPREADSHEET IN SOLVING POLYNOMIAL EQUATIONS

By

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ABSTRACT

The purpose of this paper is to demonstrate the use of the Quattro Pro Spreadsheet [2] to solve polynomial equations using Newton's method. The apprach uses matrix operations [3] to compute the derivative of the polynomial functions. THis approach is uique in the sene that it does not assume the knowledge of rules of differentiation. Two numerical examples explain the results.

1. INTRODUCTION

Computer spreadsheets such as Lotus 1-2-3 and Quattro Pro have many uses in addition to their obvious uses as spreadsheets. Their built-in mathematical functions and graphing capabilities have enhanced their usefulness to the mathematical world. Programming in a spreadsheet using macros and the built in functions is both exciting and powerful.

In this article we explain how we used Quattro Pro to write a program that uses the iterative Newton Raphson method to find the roots of any polynomial functions of degree five or less. On entering the coefficients of a polynomial, the computer shows a graph of the polynomial. One has to examine the graph to determine the first estimate of a root On feeding the estimated root, the computer (using Newton-Raphson's method) returns the value of a root correct to eight places. Graph is used again to estimate additional roots. At the end of the program the computer displays the polynomial, the roots that were found and a graph of the polynomial.

One of the authors has developed a unique method for evaluating the iterative values of Newton-Raphson's method using matrices [3]. The fact that the use of matrices that makes programming with macros very straight- forward is explained later in this paper.

The program is designed for students who are finding roots of equations for the first time. At the other extreme, roots found by Newton-Raphson's method can be compared. The next section explains the mathematics and matrix implementation of the program.

2. NEWTON-RAPHSON METHOD

Finding roots of polynomial functions is a very important problem in algebra and numerical analysis. There is a strong relationship between graphical and algebraic approaches. The iterative method is used here to find approximate roots of the polynomial equation of the form $P(x) = 0$, where

$$P(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0 y$$

is a polynomial of degree m. The method begins with an initial approximation x_0 to the root of the equation. Then subsequent approximations $x_1, x_2, \ldots$ are generated using the formula

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)} \text{ for } n > 0,$$

where $P'(x_n)$ denotes the derivative of $P(x)$ at x_n and is assumed to be non-zero.

The process terminates when two successive values of x are within a preset value of each other.

3. MATRIX IMPLEMENTATION

When using a computer, it is easy to use the matrix approach for finding the value of a function and its derivative. The matrix approach can be explained as : Let F be the set of all differentiable polynomial functions in variable x and M be the set of all 2×2 nonzero matrices [4].

Let $\$: F \to M$ be a map such that

$$\$P = \begin{bmatrix} P & P' \\ 0 & P \end{bmatrix},$$

where P' is the derivative of P. Then this map preserves the respective operations of addition and multiplication in F and M.

$$\$x = \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}$$

and
$$\$x^2 = \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix} \cdot \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}, \text{ etc.}$$

Newton-Raphson's method requires the values of P_n and P'_n of the polynomial function and its derivative for each approximation x_n. These values can be directly evaluated using the corresponding matrix for the polynomial function.

Therefore,

$$\$P|_{x_n} = \begin{bmatrix} P_n & P'_n \\ 0 & P_n \end{bmatrix}$$

and thus for

$$x_{n+1} = x_n - P_n/P'_n$$

the next approximation is

$$\$P|_{x_{n+1}} = \begin{bmatrix} P_{n+1} & P'_{n+1} \\ 0 & P_{n+1} \end{bmatrix}, \text{ etc.}$$

4. EXAMPLES

Example 1. Consider the polynomial equation

$$p(x) = x^3 - x - 1 = 0 \tag{i}$$

Since $P(1) = -1 < 0$ and $P(2) = 5 > 0$, the solution lies between 1 and 2.

Let $x_1 = 1$ be our initial approximation.

$$\$P|_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$$

and

$$x_2 = 1 - \left(\frac{-1}{2}\right) = 1.5 \tag{ii}$$

and

$$\$P|_{1.5} = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}^3 - \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .875 & 5.75 \\ 0 & .875 \end{bmatrix}$$

and

$$x_3 = 1.5 = \left(\frac{.875}{5.75}\right) = 1.34784 \tag{iii}$$

Again

$$\$P|_{1.34783} = \begin{bmatrix} .10070 & 4.44994 \\ 0 & .10070 \end{bmatrix}$$

and

$$x_4 = 1.34783 - \frac{.10070}{4.44994} = 1.32520 \tag{iv}$$

For the next approximation

$$\$P|_{1.32520} = \begin{bmatrix} .00206 & 4.26847 \\ 0 & .00206 \end{bmatrix}$$

and

$$x_4 = 1.3252 - \frac{.00206}{4.26847} = 1.32472 \tag{v}$$

$$\$P\,|_{1.32472} = \begin{bmatrix} .00001 & 4.26465 \\ 0 & .00001 \end{bmatrix}$$

and

$$x_5 = 1.32472 - \frac{.00001}{4.26465} \tag{vi}$$

$$= 1.32472$$

The process terminates, approximations x_4 and x_5. are identical. Therfore, the solution is approximately $x \approx 1.3247$.

Example 2. Let

$$P(x) = 5x^5 - 7x^4 + 6x^3 + 2x^2 - x + 8$$

be the polynomial under consideration. The user enters the coefficients 5, – 7, 6, 2, – 1, 8 as prompted by the computer. The initial spreadsheet is shown in Figure 1. (This part of the spreadsheet does not appear on the screen for the user.)

```
             A                            B

 1   8.000000000000000          0.000000000000000
 2   0.000000000000000          8.000000000000000

 4  -1.000000000000000          0.000000000000000
 5   0.000000000000000         -1.000000000000000

 7   2.000000000000000          0.000000000000000
 8   0.000000000000000          2.000000000000000

10   6.000000000000000          0.000000000000000
11   0.000000000000000          6.000000000000000

13  -7.000000000000000          0.000000000000000
14   0.000000000000000         -7.000000000000000

16   5.000000000000000          0.000000000000000
17   0.000000000000000          5.000000000000000
```

Figure 1

where the constant 8 is stored in matrix $A1 \dots B2 : -1$, the coefficient of x, is stored in matrix $A4 \dots B5$; 2, the coefficient of x^2, is stored in matrix $A7 \dots B8$; 6 the coefficient of x^3 is stored in matrix $A_{10} \dots B_{11}$; -7 the coefficient of x^4 is stored in matrix $A_{13} \dots B_{14}$; 5 the coefficient of x_n^5, is stored in matrix $A_{16} \dots B_{17}$.

The user then has the opportunity to have the computer graph for this polynomial. He/she enters the left- and right-range values for the variable x.

The computer uses these ranges to compute 100 ordered pairs of points and plots these points on a rectangular coordinate system. The graph of $p(x)$ for x in $[-1.5, 1.5]$ is shown in Figure 2.

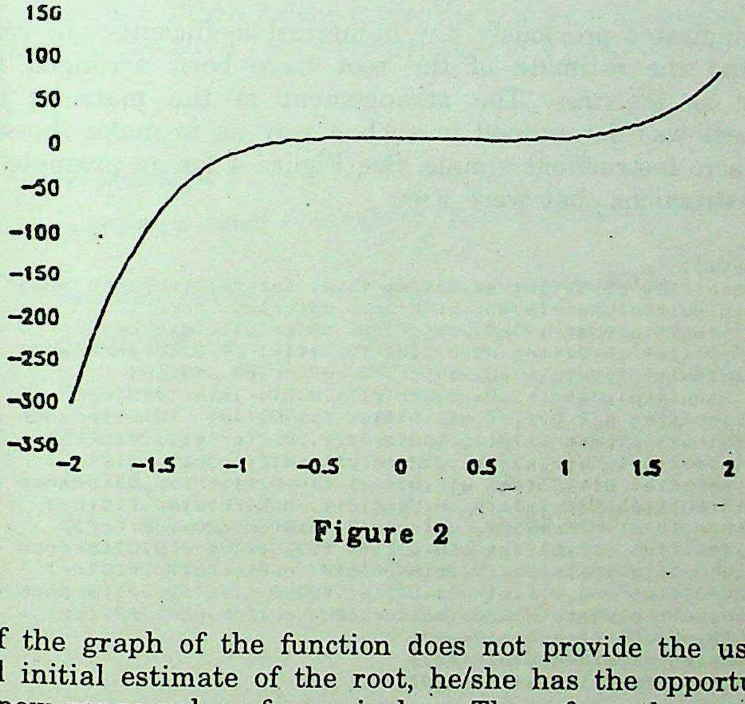


Figure 2

If the graph of the function does not provide the user with a good initial estimate of the root, he/she has the opportunity to enter new range values for varivale x. Then, from the new graph, users can choose an appropriate initial esitmate of root. The user can repeat this process to obtain a reasonable first estimate of root.

The user enters his/her first estimate at the computer prompt. If a user entered -1 as his/her first estimate it is sotred in form of matrix in $C4 \dots D5$

The information needed for the computation is arranged in matrices in the spreadsheet shown in Figure 3. Matrix $C1 \dots D2$ is an identity matrix.

```
                                                            1.000000000000000     0.000000000000000
 1  8.000000000000000        0.000000000000000              0.000000000000000     1.000000000000000
 2  0.000000000000000        8.000000000000000
 3                                                         -1.000000000000000     0.000000000000000
 4 -1.000000000000000        0.000000000000000              0.000000000000000    -1.000000000000000
 5  0.000000000000000       -1.000000000000000
 6
 7  2.000000000000000        0.000000000000000
 8  0.000000000000000        2.000000000000000
 9
10  6.000000000000000        0.000000000000000
11  0.000000000000000        6.000000000000000
12
13 -7.000000000000000        0.000000000000000
14  0.000000000000000       -7.000000000000000
15
16  5.000000000000000        0.000000000000000
17  0.000000000000000        5.000000000000000
```

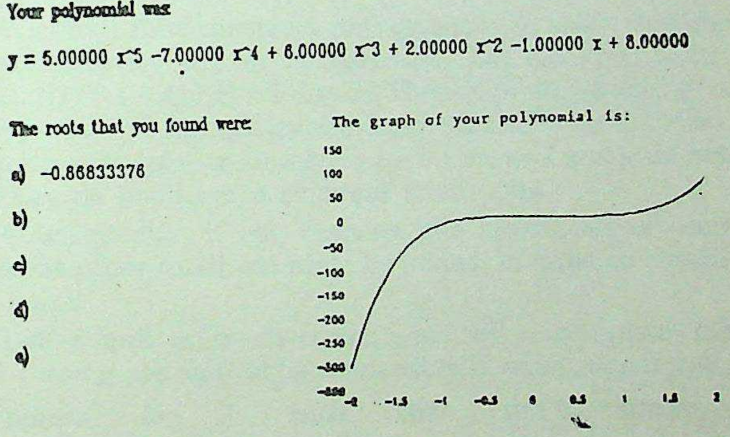
Figure 3

As indicated previously, the numerical coefficients, the constant terms, and the estimate of the root have been arranged in the computer as matrices. The arrangement of the matrices in the spreadsheet was determined in such a way as to make the writing of the macro instructions simple. See Figure 4 for an example of the macro instructions that were used.

```
83  {let c9,0}
84  {if a5=0}{let e5,0}{let e6,0}{let f5,0}{let f6,0}{branch a86}
85  {/ math;multiplyMatrix}a5..b6{cr}c5..d6{cr}e5..f6{cr}
86  {/ math;multiplyMatrix}c8..d9{cr}c8..d9{cr}c11..d12{cr}
87  {if a8=0}{let e8,0}{let e9,0}{let f8,0}{let f9,0}{branch a89}
88  {/ math;multiplyMatrix}a8..b9{cr}c8..d9{cr}e8..f9{cr}
89  {/ math;multiplyMatrix}c8..d9{cr}c11..d12{cr}c14..d15{cr}
90  {if a11=0}{let e11,0}{let e12,0}{let f11,0}{let f12,0}{branch a92}
91  {/ math;multiplyMatrix}a11..b12{cr}c11..d12{cr}e11..f12{cr}
92  {/ math;multiplyMatrix}c11..d12{cr}c11..d12{cr}c17..d18{cr}
93  {if a14=0}{let e14,0}{let e15,0}{let f14,0}{let f15,0}{branch a95}
94  {/ math;multiplyMatrix}a14..b15{cr}c14..d15{cr}e14..f15{cr}
95  {/ math;multiplyMatrix}c8..d9{cr}c17..d18{cr}c20..d21{cr}
96  {if a17=0}{let e17,0}{let e18,0}{let f17,0}{let f18,0}{branch a98}
97  {/ math;multiplyMatrix}a17..b18{cr}c17..d18{cr}e17..f18{cr}
98  {if a20=0}{let e20,0}{let e21,0}{let f20,0}{let f21,0}{branch a100}
99  {/ math;multiplyMatrix}a20..b21{cr}c20..d21{cr}e20..f21{cr}
100 {let e23,e5+e8+e11+e14+e17+e20}
101 {le[illegible]e24,e6+e9+e12+e15+e18+e21}
102 {let f23,f5+f8+f11+f14+f17+f20}
```

Figure 4

Notice the pattern that appears in the cell labels. This is one illustration of the power of macros as a programming tool. The entire program is written as a series of macros that are executed one after another. Using a spreadsheet and macros enables the programmer to use the functions that are build into the spreadsheet rather than recreating them. After the multiplications and additions have been completed, the diference between the last two approximations of the root is computed. If this difference is less than a preset value, the aproximation for the root is printed out. If the difference is greater than the preset value (.00001), the program loops to the beginning of the series of macros and repeats the process. The computer tells the user when each iteration has been completed. The maximum number of iterations has been set to twenty. If a root has not been found after twenty iterations, the computer suggests to the user that he/she should try a different number as an esitmate of the root. After a root has been determined, the user has the option to find another root, or to quit. When he/she quits, the computer displays the original polynomial, the roots that were found, and a graph of the polynomial.



```
Your polynomial was:

y = 5.00000 x^5 -7.00000 x^4 + 6.00000 x^3 + 2.00000 x^2 -1.00000 x + 8.00000

The roots that you found were:          The graph of your polynomial is:

a) -0.86833376

b)

c)

d)

e)
```

This program can be used by teachers at several different levels.

Algebra teachers discussing roots of polynomials could use the program to show students the relationship between the graphs of the polynomials and the value of the roots of the polynomials. Since the numerical computations never appear on the screen, students do not need to understand the method being used to determine the roots. Students could use the program in a computer laboratory to develop an uderstanding of the relationships that exist. They could examine various types of polynomials and discover many interesting and important ideas. This use of the program could range from elementary algebra through pre-calculus courses.

REFERENCES

[1] Howard Anton, *Calculus with Analytic Geometry*, Third eddition, John Wiley Sons, New York, 1988.

[2] Borland International, Quattro Pro, 1990.

[3] V. Chadha Differentiating polynomial/rational functions with matrices, Mathematics in College. 1990, 44-47.

[4] I.N. Herstein, *Topics in Algebra*, Second edition, Xerox, Lexington, Massachusetts, 1975.

[5] V.S. Ramaurthi, Spreadsheet works : finding maximum and minimum of functions without calculus, *J. Comput. Mat. Sci. Teach.* 7 no. 4, (1988), 81-83.

Jñānābha, Vol. 23, 1993

(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

AN EXPANSION MAPPING THEOREM

By

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In a recent paper [1], Daffer and Kaneko proved the following fixed point theorem :

Theorem ([4], Theorem 3.) *Let (X, d) be a complete metric space. Let f be a surjective selfmap of X and g and injective selfmap of X which satisfy the following condition : there exists a number $q > 1$ such that*

$$d(fx, fy) \geq qd(gx, gy) \tag{1}$$

for each $x, y \in X$. If f and g commute, then there exists a unique common fixed point of f and g.

We shall first construct and example to show that the condition of commutativity cannot be removed. Let $X = [0, 1]$ with the usual metric, $f(x) = 1 - x$, $g(x) = 1 - x/2$. Then f is surjective, g is injective, $d(fx, fy) = |x - y| \geq qd(gx, gy)$ for $1 < q < 2$ and (1) is satisfied. However, for each $x \in X$, $fgx = \varepsilon x$ and $gfx = 1 - \varepsilon + \varepsilon x$, so f and g do not commute. Also they do not have a common fixed point.

Consequently, if one removes the hypothesis of commutativity, then some other condition must be added, in order to obtain a common fixed point.

Let f and g be two selfmaps of a complete metric space (X, d). f and g are said to be compatible if, whenever $\{x_n\}$ is a sequence of points in X such that $\lim fx_n = \lim gx_n = t$, then $\lim d(fgx_ngfx_n) = 0$.

The following result is an extension of the above Theorem to compatible maps :

Theorem. Let (X, d) be a complete metric space, f and g compatible selfmaps of X satisfying conditions (1) and $g(X) \subseteq f(X)$, f continuous. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, choose x_1 such that $fx_1 = gx_0$. In general, choose x_{n+1} such that $fx_{n+1} = gx_n$. Then, from (1),

$$d(gx_n, gx_{n+1}) \leq d(fx_n, fx_{n+1})/q = d(gx_{n-1}, gx_n)/q,$$

which implies that $\{gx_n\}$ is Cauchy, hence convergent. Call the limit z. Then $\lim gx_n = \lim fx_n = z$. Since f is continuous. $\lim fx_n = fz$. Since f and g compatible, $\lim d(gfx_n, fgx_n) = 0$, which, since $\lim fgx_n = fz$, implies that $\lim gfx_n = fz$.

From (1),

$$d(gfx_n, gx_n) \le d(f^2x_n, fx_n)/q.$$

Taking the limit as $n \to \infty$ yields

$$d(fz, z) \le d(fz, z)/q,$$

which implies that $z = fz$.

Again from (1),

$$d(gx_n, gz) \le d(fx_n, fz)/q.$$

Taking the limit as $n \to \infty$ gives $d(z, gz) \le d(z, fz)/q = 0$ and $z = gz$.

Suppose that w is also a common fixed point of f and g. Then

$$d(z, w) = d(gz, gw) \le d(fz, fw)/q = d(z, w)/q,$$

which implies that $z = w$, and the common fixed point is unique.

REFERENCE

[1] Peter Z. Daffer and Hideaki Kaneko, On expansive mappings, *Math. Japonica* **37** (1992), 733-735.

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(Dedicated to Professor S.A. Naimpally on his 62th Birthday)

FIXED POINT THEOREM FOR THREE MAPPINGS ON METRIC SPACE

By

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ABSTRACT

Bhagwat and Singh [1] have extended the result of Das and Gupta [2] for pair of mappings. In this paper we prove a fixed point theorem for three mappings on a metric space, giving a new type of rational inequality like that of Bhagwat and Singh [1].

Definition : Let S, T and P be three self mappings of a metric space (X,d), then we say that $\{S, T\}$ is a weakly commuting pair of mappings with respect to mapping p,
if $d(PSPx, TPx) < d(SPPz, TPx)$ (i)
and $d(SPx, PTPx) < d(SPx, TPPx)$ (ii)
for all x in X.

Now we establish the following

Theorem : Let (x, d) be a complete metric space, let P,S and T be continuous self mappings on x such that $\{S, T\}$ is a weakly commuting pair of mappings with respect to mapping P and

$$d(SPx, TPy) \leq \frac{d(x, SPx)\, d(y, TPy) + d(y, SPx) d\,(x, TPy)}{d(x, SPx) + d(x, TPy)}$$

for alll x, y in X. If for some $x_0 \in X$, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to a point $z \in X$, then z is a common fixed point of SP and is a unique common fixed point of S,P and T.

Proof : Let $x_0 \in X$ such that

$$x_{2n+1} = SPx_{2n},\ x_{2n} = TPx_{2n-1},\ x_{2n+2} = TPx_{2n+1}\,.$$

we have

$$d(x_{2n+1}, x_{2n+2}) = d(SPx_{2n}, TPx_{2n+1})$$

$$\leq \frac{d(x_{2n}, SPx_{2n})\, d(x_{2n+1}, TPx_{2n+1}) + d(x_{2n+1}, SPx_{2n})\, d(x_{2n}, TPx_{2n+1})}{d(x_{2n}, SPx_{2n} + d(x_{2n}, TPx_{2n+1})}$$

$$= \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}), x_{2n+1}\, d(x_{2n}, x_{2n+2})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})}$$

$\leq d(x_{2n}, x_{2n+1})$.

Continuing in this way we obtain

$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) = \ldots \leq d(x_0, x_1)$. So we have a monotonic sequence of positive real numbers.

Let the monotonic sequence converges to a real number l. Since $\{x_n\}$ has convergent subsequence $\{x_{n_k}\}$ in X which converges to some $z \in X$, therefore

$$\lim_{n \to \infty} x_{2n_k} = z.$$

Then we show that z is a common fixed point of SP and TP. If it is not possible we assume that $z \neq SPz$.

Now

$$\begin{aligned} d(z,SPz) &= d(\lim x_{2n_k}, SP\, lim\, x_{2n_k}) \\ &= \lim d(x_{2n_k}, SPx_{2n_k}) \\ &= \lim d(x_{2n_k}, x_{2n_{k+1}}) \\ &= \lim d(x_{2n_{k+1}}, x_{2n_{k+2}}) \\ &= \lim d(SPx_{2n_k}, TPSPx_{2n_k}) \\ &= d(\lim SPx_{2n_k}, \lim TPSPx_{2n_k}) \\ &= d(SPz, TPSPz) \\ &\leq \frac{d(z, SPz)\, d(SPz, TPSPz) + d(SPz, SPz)\, d(z, TPSPz)}{d(z, sPz) + d(z, TP\, SPz)} \end{aligned}$$

i.e. $d(z, SPz) \leq d(z, SPz)$

which is a contradiction. Therefore $z = SPz$, so that z is a fixed point of SP.

In the same way let $z \neq TPz$, we get

$$d(z, TPz) = d(SPz, TPSPz). \leq d(z, SPz) = 0$$

a contradiction. Hence z is a fixed point of TP also. Thus

$$SPz = z = TPz.$$

Now we shall show that z is a common fixed point of S,P and T. Using (i) we have

$$\begin{aligned} d(Pz, z) &= d(PSPz, TPz) \\ &< d(SPPz, TPz) \\ &\leq \frac{d(Pz, Pz)\, d(z, z) + d(z, Pz)\, d(Pz, z)}{d(Pz, Pz) + d(Pz, z)} \end{aligned}$$

i.e., $$d(Pz, z) \leq d(Pz, z).$$

This implies that $Pz = z$. Hence by (i) we have $Sz = z\ Tz$. Similarly using condition (ii) we have

$$Pz = z = Sz = Tz.$$

Therefore z is a common fixed of S, T and P. For the uniqueness of fixed point, let z and z' be the two common fixed points of S, P and T. Then

$$d(z, z') = d(SPz, TPz')$$

$$\leq \frac{d(z, SPz)\, d(z', TPz') + d(z', SPz) d(z, SPz')}{d(z, SPz) + d(z, SPz')}$$

i.e. $d(z, z') \leq d(z, z')$,

which is a contradiction, therefore z is unique common fixed point of S,P and T.

This completes the proof of the theorem

REFERENCES

[1] Bhagwat and Singh, Fixed point of a pair of mapping, *Indian J. Pure. Appl. Math.* **17**(8), (1986), 994-997.

[2] Dass and Gupta, *Indian J. Pure Appl Math.* **6**, (1975), 1455-1458.

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